Auto-Determining Mainlobe Width for Beampattern Synthesis via Relaxation Optimization

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***Abstract*—This paper presents a novel array pattern synthesis algorithm based on relaxation optimization. We consider the problem of how to auto-determine the width of mainlobe region in beampattern synthesis. By introducing a slack region and a slack vector, the beampattern synthesis problem can be transformed as a relaxation optimization problem. The relaxation optimization problem is non-convex with an objective of sparsifying the slack vector. The sequential convex optimization procedure is then applied to solve the relaxation optimization problem and find its sparse solution. Representative simulations are provided to demonstrate the effectiveness of the proposed method in beampattern synthesis.**

***Index Terms*—Beampattern synthesis, relaxation optimization, mainlobe width, sparse optimization.**

I. Introduction

ARRAY antenna has found numerous applications to radar, navigation, wireless communications and other fields [1]–[5]. Optimal array antenna design plays a critical role in improving system performance and reducing cost. Over the past several decades, quite a number of approaches to pattern synthesis have been developed. The classical algorithms [6]–[8] have closed-form expressions but they are limited to some specific array geometries or array patterns. Global optimization-based methods like genetic algorithm (GA) [9] and simulated annealing (SA) method [10] search the optimal solutions using stochastic approaches. However, these methods are time-consuming in beampattern design.

A different class of algorithms for array synthesis have been devised using matrix pencil technique; see the work in [11]–[14] for details with various applications. In addition, compressive sensing (CS) has been applied to array synthesis with high efficiency and reduced computational cost [15]–[18]. Apart from the aforementioned methods, there also exist a few approaches attempting to synthesize patterns by utilizing the least-squares method [19] or the excitation matching approach [20].

More recently, array pattern synthesis problem has been devised along with the advances in convex optimization [21]. For instance, the authors of [22] propose to design optimal beampatterns with the aid of convex programming (CP). To formulate the non-convex lower bound constraints as convex ones, the conjugate symmetric weights are utilized in [23] to design beampatterns for linear and planar arrays. With the idea of semidefinite relaxation (SDR) in [24], the authors of [25] propose an effective beampattern synthesis algorithm with upper and lower constraints. Other convex optimization-based methods can be found in [26]–[28]. Nevertheless, the convex optimization-based beampattern synthesis algorithms have to empirically prescribe the width of mainlobe region. For a non-regular array (such as nonisotropic conformal array), an inappropriate setting on mainlobe width may lead to a distorted beampattern or an unsolvable result.

The drawback of existing work motivates us to develop a new beampattern synthesis algorithm in this paper. The proposed algorithm can auto-determine the mainlobe width in beampattern synthesis. We introduce a slack region for beampattern synthesis and formulate a relaxation optimization problem to minimize the mainlobe width. On this basis, the sequential convex optimization procedure is applied to find the ultimate weight vector. The proposed algorithm has no limitation on the array configuration and pattern shape. It is applicable to synthesize focused and shaped patterns for two-dimensional or conformal array.

II. Problem Formulation

Let us consider an N-element antenna array with arbitrary geometry. For the sake of clarity, the problem is described for a one-dimensional case. The steering vector at direction θ is given by

$$\mathbf{a}(\theta) = \left[g_1(\theta)e^{j\phi_1(\theta)}, \ldots, g_N(\theta)e^{j\phi_N(\theta)}\right]^{\mathrm{T}} \tag{1}$$

where $g_n(\theta)$ represents the radiation pattern of the nth element, $\phi_n(\theta)$ stands for the phase delay of the nth element, $n = 1, \ldots, N$. Then, the magnitude of the array response is given by

$$f(\theta) = |\mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta)| \tag{2}$$

where $\mathbf{w} = [w_1, w_2, \ldots, w_N]^{\mathrm{T}}$ is the weight vector, $(\cdot)^{\mathrm{H}}$ denotes conjugate transpose operator, θ_0 stands for the beam axis.

Let θ_0 be the beam axis and denote the sidelobe region as Φ_{side}. For focused beampattern synthesis, the conventional CP method pre-assign a mainlobe region $\Phi_{\mathrm{main}} = [\theta_0 - \alpha, \theta_0 + \alpha]$, where α is a real constant that is usually set empirically. Then, pattern synthesis can be formulated as a convex optimization problem as

$$\text{find } \mathbf{w} \tag{3a}$$

$$\text{s.t. } \mathbf{w}^H\mathbf{a}(\theta_0) = 1 \tag{3b}$$

$$|\mathbf{w}^H\mathbf{a}(\theta_i)| \leq \rho(\theta_i),\ \theta_i \in \Phi_{\mathrm{side}} \tag{3c}$$

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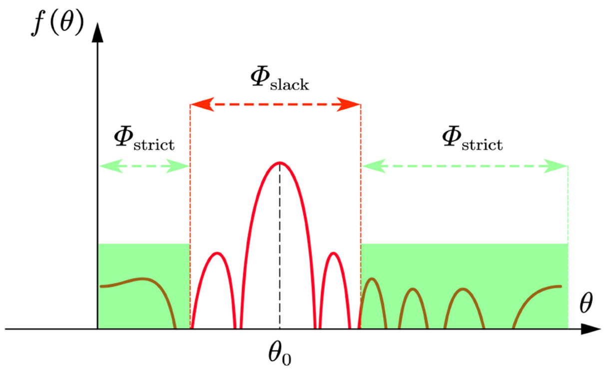


Fig. 1. Illustration of Φ_{slack} and Φ_{strict}.

Since Φ_{side} depends on the setting of Φ_{main}, an inappropriate α may lead to a distorted beampattern or an unsolvable result for the above problem. In the next section, we present a new algorithm that can auto-determine the mainlobe width in beampattern synthesis.

III. The Proposed Method

In this section, we formulate the beampattern synthesis problem as a relaxation optimization problem. Then, a sequential convex optimization procedure is applied to find the ultimate solution.

A. Relaxation Optimization Formulation for Beampattern Synthesis

To auto-determine the mainlobe width in beampattern synthesis, we define a slack region as

$$\Phi_{\text{slack}} = [\theta_0 - \beta, \theta_0 + \beta] \tag{4}$$

where β is positive that can be set flexibly. Accordingly, a strict region is denoted as Φ_{strict}. We have $\Phi_{\text{slack}} \cap \Phi_{\text{strict}} = \emptyset$ and $\Phi_{\text{slack}} \cup \Phi_{\text{strict}} = \Phi$, as shown in Fig. 1. In Φ_{strict}, strict constraints are imposed to restrict the magnitude of pattern to be lower than the given mask $\rho(\theta)$. In Φ_{slack}, a slack vector (denoted as $\mathbf{s}$) is introduced to allow the beampattern be higher than the sidelobe mask. To have an effective relaxation, we constrain $\mathbf{s}$ as

$$\mathbf{s} \succeq \mathbf{0} \tag{5}$$

Then, a relaxed formulation for pattern syntheis can be written as

$$\text{find } \mathbf{w} \tag{6a}$$
$$\text{s.t. } \mathbf{w}^H \mathbf{a}(\theta_0) = 1 \tag{6b}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_l)| \leq \rho(\theta_l) + s_l,\ \theta_l \in \Phi_{\text{slack}} \tag{6c}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_i)| \leq \rho(\theta_i),\ \theta_i \in \Phi_{\text{strict}} \tag{6d}$$

where s_l is the element of $\mathbf{s} = [s_1, s_2, \ldots, s_L]^{\mathrm{T}}$, L stands for the length of $\mathbf{s}$. It can be seen that the above problem (6) degenerates into the problem (3) when $\mathbf{s} = \mathbf{0}$.

To synthesize a satisfactory beampattern, we further constrain the monotonicity of $\mathbf{s}$ to be consistent with the curve of a desired mainlobe. More specifically, we constrain $\mathbf{s}$ to be monotonically nondecreasing when $\theta \in [\theta_0 - \beta, \theta_0]$, and constrain $\mathbf{s}$ to be monotonically nonincreasing when $\theta \in [\theta_0, \theta_0 + \beta]$. These constraints can be expressed compactly as

$$\mathbf{G}\mathbf{s} \preceq \mathbf{0} \tag{7}$$

where the matrix $\mathbf{G}$ has the following form

$$\mathbf{G} = \begin{bmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V} \end{bmatrix} \in \mathbb{R}^{(L-2)\times L} \tag{8}$$

and the matrix $\mathbf{V}$ is shown as follow

$$\mathbf{V} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(\frac{L}{2}-1)\times \frac{L}{2}} \tag{9}$$

where L is implicitly assumed to be even.

Since fewer non-zero component in $\mathbf{s}$ implies a narrower mainlobe, one can minimize the mainlobe width by sparsifying $\mathbf{s}$. Thus, the following relaxation optimization problem can be formulated to auto-determine the mainlobe width as

$$\min_{\mathbf{w},\mathbf{s}} \ ||\mathbf{s}||_0 \tag{10a}$$
$$\text{s.t. } \mathbf{w}^H \mathbf{a}(\theta_0) = 1 \tag{10b}$$
$$\mathbf{s} \succeq \mathbf{0} \tag{10c}$$
$$\mathbf{G}\mathbf{s} \preceq \mathbf{0} \tag{10d}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_l)| \leq \rho(\theta_l) + s_l,\ \theta_l \in \Phi_{\text{slack}} \tag{10e}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_i)| \leq \rho(\theta_i),\ \theta_i \in \Phi_{\text{strict}} \tag{10f}$$

To further solve the non-convex problem (10), a sequential convex optimization procedure is applied as presented next.

B. Solving Problem (10) Via Sequential Convex Optimization

A sequential convex optimization approach is introduced next to solve the problem (10). We first define a penal vector $\mathbf{p} = [p_1, p_2, \ldots, p_L]^{\mathrm{T}}$, and adopt the following V-shaped curve for its initialization:

$$p_l = \begin{cases} L - l, & \text{for } l = 1, \ldots, \frac{L-1}{2} \\ l, & \text{for } l = \frac{L}{2}, \ldots, L \end{cases} \tag{11}$$

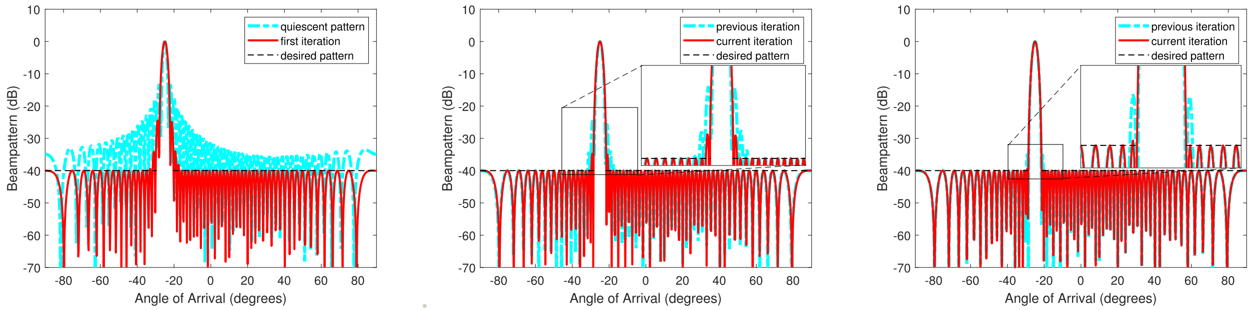
With the above $\mathbf{p}$, a sub-optimal solution to the problem (10) can be obtained by solving the following convex optimization problem as

$$\min_{\mathbf{w},\mathbf{s}} \ \mathbf{p}^H \mathbf{s} \tag{12a}$$
$$\text{s.t. } \mathbf{w}^H \mathbf{a}(\theta_0) = 1 \tag{12b}$$
$$\mathbf{s} \succeq \mathbf{0} \tag{12c}$$
$$\mathbf{G}\mathbf{s} \preceq \mathbf{0} \tag{12d}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_l)| \leq \rho(\theta_l) + s_l,\ \theta_l \in \Phi_{\text{slack}} \tag{12e}$$
$$|\mathbf{w}^H \mathbf{a}(\theta_i)| \leq \rho(\theta_i),\ \theta_i \in \Phi_{\text{strict}} \tag{12f}$$

To enhance the sparsity of $\mathbf{s}$, a sequential convex optimization procedure is conducted. More specifically, we update the vector $\mathbf{p}$ at the iteration k as

$$\mathbf{p}^k = \mathbf{1} \oslash \mathbf{s}^{k-1} \tag{13a}$$
$$\mathbf{s}^{k-1} = \chi(\bar{\mathbf{s}}^{k-1}) \tag{13b}$$

where $\oslash$ denotes the element division operation and χ removes the zero components of a vector, $\mathbf{s}^{k-1}$ is a vector that only contains the non-zero components of $\bar{\mathbf{s}}^{k-1}$. We conduct the above update iteratively and stop the iteration if a stopping criteria is satisfied. It should be emphasized that we remove zeros



(a) (b) (c)

Fig. 2. Simulation results of pattern synthesis of uniform sidelobe. (a) The synthesized beampattern at the first step. (b) The synthesized beampattern at the second step. (c) The synthesized beampattern at the third step.

TABLE I
SUMMARY OF THE PROPOSED PATTERN SYNTHESIS ALGORITHM

Input	θ_0, $\mathbf{a}(\theta)$, $\rho(\theta)$, β, Φ_{slack}
Step 1.	(1). Calculate the length of Φ_{slack}, L.
	(2). Initialize $\mathbf{G}$ using (8) and (9).
	(3). Initialize $\mathbf{p}$ using (11).
Step 2.	Solve problem (12).
Step 3.	(1). Update $\mathbf{p}$ and $\mathbf{s}$ using (13).
	(2). Using the new $\mathbf{s}$ to update Φ_{slack}.
Step 4.	(1). Using $\mathbf{w}$ to obtain the beampattern.
	(2). Compute η.
Step 5.	Go to **Step 2.** unless η is small enough.
Output	$\mathbf{w}$.

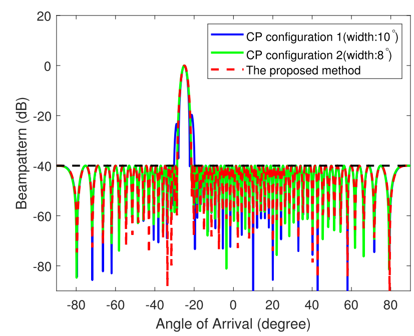


Fig. 3. Result comparison with CP method.

from both ends of the vector $\bar{\mathbf{s}}^{k-1}$. Although the convergence may not be proved from theoretical perspective, extensive simulations show that our algorithms converge well under different situations.

To verify the performance of the proposed algorithm, we define the following η as

$$\eta = \int_{\mathbb{D}} \max\{P(\theta) - \rho(\theta), 0\} \mathrm{d}\theta \tag{14}$$

where $\mathbb{D} = [-\pi/2, \theta_L] \cup [\theta_R, \pi/2]$, θ_L and θ_R are the first-nulls on the left and right side of the obtained pattern, respectively. In our algorithm, we can stop the iteration when η is small enough. Finally, the proposed pattern synthesis algorithm is summarized in Table I.

Remark 1: In (11), we adopt a V-shaped curve as the initialization for $\mathbf{p}$, for the reason that it penalizes heavily on both ends of vector $\mathbf{s}$. This is a simple and effective way to generate sparse $\mathbf{s}$. In fact, different initializations for $\mathbf{p}$ can be adopted and the ultimate pattern synthesis results are almost identical. The reason is that, the intermediate result for $\mathbf{p}$ in the current iteration depends on the resulting $\mathbf{s}$ of the previous iteration. With the monotonicity constraint on $\mathbf{s}$ (which is a strong constraint) and after enough iterations, the influence of the initial setting for $\mathbf{p}$ becomes weak for the ultimate result.

IV. NUMERICAL RESULTS

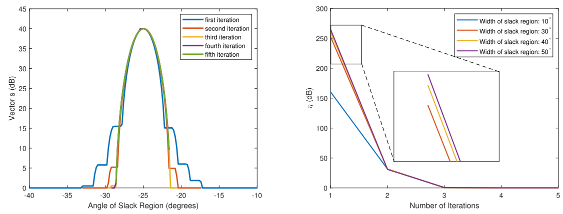
In this section, several representative simulations are provided to demonstrate the effectiveness of the proposed method. Firstly, a uniform linear array (ULA) with uniform sidelobe is considered. Secondly, we consider beampattern synthesis for a nonuniformly spaced linear array. Finally, the performance of the proposed algorithm for a nonisotropic array is investigated.

A. Uniform Linear Array

In the first example, a ULA with 60 elements is considered. The interval between each element is half of the wavelength and the beam axis is fixed at $\theta_0 = -25^\circ$. We take $\beta = 15^\circ$ and $\Phi_{\text{slack}} = [-40^\circ, -10^\circ]$. The upper bound of the sidelobe is -40 dB.

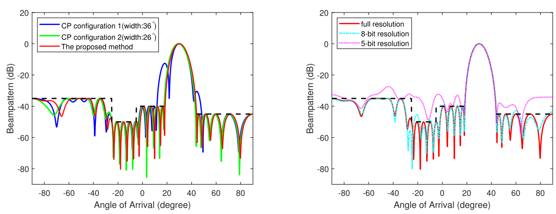
1) Beampattern Results Comparison: With the above configurations and after five iteration steps, the proposed algorithm can synthesize a desirable beampattern. Fig. 2 compares the pattern results at different iterations. Fig. 2(a) shows the quiescent pattern and the pattern resulted after the first iteration. It can be seen that the sidelobe of the proposed algorithm outside the slack region perfectly meets the requirement of the desired pattern. Nevertheless, in the slack region especially in $[-30^\circ, -15^\circ]$, the sidelobe level is greater than the desired one. Fig. 2(b) compares the beampatterns obtained at the first and the second iterations, and Fig. 2(c) shows the beampattern at the 3 rd iteration. One can see that the sidelobe around the beam axis become lower as the iteration continues. After 4 iterations, our algorithm can synthesize a satisfactory result. The running time of the proposed algorithm is about 60 s.

Fig. 3 compares the ultimate beampattern of the proposed algorithm and the result obtained by CP method. For CP method, different settings (10°, 8° and 6°, respectively) for mainlobe width are considered and tested. From Fig. 3, one can see that the mainlobe of CP method is too wide and has been distorted when the width is taken as 10°. In the case when setting the mainlobe width as 8°, the CP method obtains similar result to that of the



(a) (b)

Fig. 4. Performance test of the proposed algorithm at different iterations. (a) Curves of $\mathbf{s}$ at different iterations. (b) Curves of η versus iteration with different widths of slack region.



(a) (b)

Fig. 5. Simulation results of pattern synthesis of nonuniform sidelobe. (a) Comparison of the proposed method and CP method. (b) Beampattern comparison with different phase shifter resolutions.

proposed algorithm. If the mainlobe width of CP method is taken as 6°, we find that the resulting problem becomes unsolvable. In this case, the constraints on the sidelobe are too strong.

2) Test on $\mathbf{S}$*:* To show the convergence of the proposed algorithm, we now present the value of vector $\mathbf{s}$ at different iterations. Fig. 4(a) compares the vector $\mathbf{s}$ in slack region of the first five iterations. It can be seen that the vector $\mathbf{s}$ converges to an ideal mainlobe shape, whose width is determined automatically.

3) Test on η*:* To investigate the beampattern synthesis performance with different settings of the initial $\Phi_{\rm slack}$, we now present the curve of η versus the iteration number by taking β as 5°, 15°, 20° and 25°, respectively. Fig. 4(b) shows the resulting curves, from which one can see that η converges quickly, and the value of the initial β has little impact on the convergence. With the above 4 settings on β, our algorithm can all reach satisfactory performance after 3 iteration steps.

B. Nonuniformly Spaced Linear Array

To show that our algorithm behaves well not only under carefully chosen array configurations, we carry out the second example by randomly selecting the element positions. In this simulation, a 20-element nonuniform spaced linear array is considered. We set the beam axis as $\theta_0 = 30^\circ$. The sidelobe level is expected to be lower than -35 dB if $\theta \in [-90^\circ, -25^\circ]$, -40 dB if $\theta \in [-25^\circ, -5^\circ]$, -50 dB if $\theta \in [5^\circ, 25^\circ]$, and -45 dB for θ in the interval $[25^\circ, 90^\circ]$.

The simulation result is shown in Fig. 5(a). The width of mainlobe region in the CP method is set as 36°, 26° and 24°, respectively. The proposed algorithm can obtain a beampattern that meets the requirement without setting the width of mainlobe region empirically. However, the CP method can only result a satisfactory pattern in the second mainlobe width configuration (i.e., 26°). The first mainlobe width setting (i.e., 36°) for CP method is too wide, and the resulting mainlobe has been distorted. For the second setting of the mainlobe width (i.e., 24°), we find that the resulting CP problem is unsolvable.

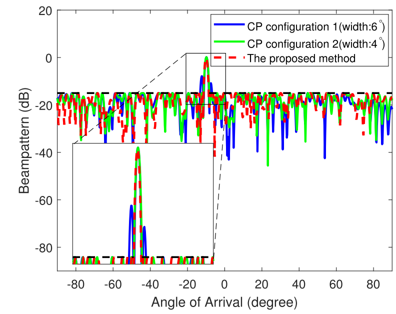


Fig. 6. Synthesized patterns with uniform sidelobe for a nonisotropic array.

To measure the performance of our method with finite-resolution phase shifters, we depict the resulting beampatterns by quantizing the phase weightings with different bits. The comparison is presented in Fig. 5(b), from which we can see that the beampattern with 8-bit phase resolution is almost identical with the full-resolution one. There is much difference between the full-resolution beampattern and the one with 5-bit phase resolution.

C. Uniform Sidelobe Synthesis for a Nonisotropic Random Array

A 40-element nonisotropic linear random array is considered next to verify the performance of the proposed algorithm. The pattern of the nth element is given by

$$g_n(\theta) = [\cos(\pi l_n \sin(\theta + \zeta_n)) - \cos(\pi l_n)] / \cos(\theta + \zeta_n) \quad (15)$$

where ζ_n and l_n represent the orientation and length of the element, respectively. More details of the array is omitted here due to the limit of space. The beam axis is $\theta_0 = -10^\circ$ and the desired pattern has a -15 dB uniform sidelobe. In this example, the mainlobe width for the CP method is taken as 6°, 4° and 2°, respectively.

Beampattern comparison between the CP method and the proposed method is depicted in Fig. 6. One can see from Fig. 6 that the proposed algorithm can synthesize a beampattern with good performance on both sidelobe and mainlobe. Similar to the previous example, the CP method can only obtain a satisfactory beampattern for the second setting (i.e., 4°) on the mainlobe width. The other two mainlobe width configurations are either too width or too narrow for the CP method. In this case, the running time of the proposed algorithm is about 52 s.

V. Conclusion

In this paper, a novel beampattern synthesis method to auto-determine the width of mainlobe region has been presented. In the proposed algorithm, a relaxation optimization problem has been developed by introducing slack variables into the conventional convex programming based beampattern synthesis formulation. We have imposed some constraints on the slack variables, and have obtained a relaxation optimization problem by sparsifying the slack variables. On this basis, a sequential convex optimization procedure has been applied to solve the relaxation optimization problem. Representative simulations have been carried out to verify the effectiveness of the proposed approach under various scenarios. The proposed method is time-consuming, and we shall consider the acceleration algorithm in our future work.

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