Synthesis of Sparse Arrays With Discrete Phase Constraints via Mixed-Integer Programming

Dongnan Xia and Xuejing Zhang, *Member, IEEE*

***Abstract*—Synthesizing sparse arrays with discrete phase constraints is a critical problem in many applications. In this letter, we present a novel approach for synthesizing sparse arrays with discrete phase constraints using mixed-integer programming (MIP). The proposed approach optimizes both antenna positions and amplitude/phase excitations subject to the given discrete phase constraints. We show that our approach is flexible and can handle various practical constrains, such as minimum spacing constraints between elements, phase-only constraints in reconfigurable arrays. For each scenario, we obtain different MIP formulations, which can be solved using off-the-shelf solvers. Representative simulations are provided to demonstrate the effectiveness and superiority of the proposed method in sparse array synthesis.**

***Index Terms*—Discrete phase constraint, mixed-integer programming (MIP), phase-only reconfigurable array, sparse array synthesis.**

I. Introduction

SPARSE antenna array is widely utilized in various fields, including wireless communication, radar, and navigation. Sparse antenna array has an advantage of reducing the hardware complexity as compared with the dense array [1], [2], [3]. Over the past few decades, numerous techniques have been developed for sparse array synthesis. As a traditional methodology for sparse array design, global optimization-based methods have been utilized to synthesize sparse arrays by finding optimal solutions through stochastic approaches. This further includes genetic algorithm [4], particle swarm optimization method [5], and simulated annealing method [6]. In general, these methods are time-consuming. Another approach for sparse array synthesis is based on matrix pencil method [7], [8], [9], [10]. However, this type of method usually requires a reference pattern and cannot accommodate additional constraints.

Recently, convex optimization techniques [11] have gained increasing popularity in the synthesis of sparse arrays. Convex optimization is applied to optimize both antenna positions and corresponding weights, to obtain the desired beampattern using the fewest elements within a given aperture [12]. For example, Fuchs [13] proposed an iterative reweighted ℓ_1 norm minimization technique to synthesize sparse arrays. This iterative reweighted method solves the convex approximation of the original ℓ_0 norm minimization problem. An effective compressed-sensing (CS) inspired deterministic algorithm is presented in [14], utilizing a modified weighting function and a novel clustering technique to reduce the computational effort and get better performance on sparse array synthesis. A reconfigurable sparse array synthesis method is proposed in [15], using focal underdetermined system solver (FOCUSS) and multiple measurement vectors (MMV) collaborative sparse recovery. A convex optimization-based synthesis algorithm with antenna selection was proposed in [16], where sparse array is designed by selecting antennas from a dense array including candidate antennas arranged closely. Moreover, the conjugate symmetric weights are utilized in [16] to maintain the convexity of the formulation. Apart from the aforementioned work, there also exist other convex optimization-based methods attempting to synthesize sparse arrays, as reported in [17], [18], [19], and [20].

The sparse array synthesis methods mentioned above do not consider discrete phase constraints for excitation. In practical applications, it is not feasible to implement continuous phase shifts for a given pattern radiation, as they require high-precision analog circuits. Instead, discrete phase shifts can provide a more feasible solution with lower implementation complexity and reduced power consumption. The work in [21], [22], [23], [24], and [25] addresses the problem of beampattern synthesis with discrete phase constraints. The methods for handling discrete phase constraints mainly include convex relaxation [21], search-type computation [22], phase correction [23], and evolutionary algorithm [24], [25]. These methods often yield suboptimal solutions. Moreover, they cannot be directly extended to consider other practical constraints, such as the minimum element spacing constraint and/or the phase-only constraint.

In this letter, drawing inspiration from the concept of antenna selection [13], [14], [15], [16], [17], [18], we find that the problem of sparse array synthesis can be reformulated as a mixed-integer programming (MIP) problem [26]. Existing work utilizing MIP to model discrete phase constraints can be found in [27] and [28]. However, these work discusses beampattern synthesis for fixed arrays. In contrast, the proposed method in this letter enables the joint design of both element positions and excitation vectors. In addition, two practical extensions are considered for the proposed method. The first one is to synthesize sparse arrays with minimum element spacing constraints [17], [18], [19], thus alleviating the mutual coupling effect between antennas. The second extension is to synthesize phase-only reconfigurable sparse arrays [29], [30], [31], [32], which is able to alter radiating characteristics without changing the amplitude excitations. For each of these scenarios, we obtain different MIP models, which can be solved using general-purpose off-the-shelf solvers, such as GUROBI [33] and CPLEX [34]. Representative examples are presented to demonstrate the effectiveness of the proposed algorithm under various situations.

Manuscript received 30 October 2023; accepted 10 December 2023. Date of publication 13 December 2023; date of current version 5 March 2024. This work was supported in part by the National Natural Science Foundation of China under Grant 62101101, in part by the Peng Cheng Shang Xue Education Fund under Grant XY2021602, and in part by the Sichuan Science and Technology Program under Grant 24NSFSC6268. *(Corresponding author: Xuejing Zhang.)*

The authors are with the School of Information and Communication Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China (e-mail: dnxiauestc@163.com; xjzhang7@163.com).

Digital Object Identifier 10.1109/LAWP.2023.3342143

II. Sparse Array Synthesis

A. Problem Description

Under the assumption of narrowband and far-field, we take the focused beampattern as an example and consider an array composed of N candidate antennas located at known positions. The antenna selection method [13] formulates sparse array synthesis problem as

$$\min_{\boldsymbol{w}} \quad ||\boldsymbol{w}||_0 \tag{1a}$$

$$\text{s.t.} \quad \Re\left\{\boldsymbol{w}^H \boldsymbol{a}(\bar{\theta}_0)\right\} = 1 \tag{1b}$$

$$\left|\boldsymbol{w}^H \boldsymbol{a}(\theta)\right| \leq \rho(\theta) \quad \forall\theta \in \mathbb{S} \tag{1c}$$

where $\boldsymbol{w} = [w_1, \ldots, w_N]^T$ is the complex excitation vector, $(\cdot)^H$ denotes the Hermitian transpose operation and $(\cdot)^T$ represents the transpose operation, $\bar{\theta}_0$ is the mainlobe axis, $\|\cdot\|_0$ represents the ℓ_0 norm, $\mathbb{S}$ stands for the sidelobe region, $\rho(\theta)$ is the upper bound for sidelobe, and $\boldsymbol{a}(\theta)$ is the steering vector defining as

$$\boldsymbol{a}(\theta) = \left[g_1(\theta)e^{-j\omega\tau_1(\theta)}, \ldots, g_N(\theta)e^{-j\omega\tau_N(\theta)}\right]^T. \tag{2}$$

In (2), $g_n(\theta)$ represents the element pattern of the nth antenna, ω is the operating frequency, $\tau_n(\theta)$ denotes the time delay between the nth element and the reference one, $n = 1, \ldots, N$.

To address the nonconvex problem (1), an iterative reweighted ℓ_1 norm minimization method is introduced in [13] to obtain an approximate solution. However, it is sensitive to initial values and lacks the flexibility to impose additional constraints. Next, we propose a new formulation for the problem of sparse array synthesis using MIP.

B. Sparse Array Synthesis Using MIP

Considering that the amplitude excitation is always limited in practical applications, it is reasonable to set the amplitude excitation not to exceed a certain threshold, i.e.,

$$|\boldsymbol{w}| \preceq \eta \cdot \mathbf{1} \tag{3}$$

where η is a sufficiently large positive constant. In (3), $\mathbf{1}$ represents a vector of all ones, whose dimension can be inferred from the context.

With the assumption (3), we can introduce a binary vector $\boldsymbol{b} = [b_1, \ldots, b_N]^T$ and reformulate the problem (1) as

$$\min_{\boldsymbol{w},\boldsymbol{b}} \quad \mathbf{1}^T\boldsymbol{b} \tag{4a}$$

$$\text{s.t.} \quad \Re\left\{\boldsymbol{w}^H \boldsymbol{a}(\bar{\theta}_0)\right\} = 1 \tag{4b}$$

$$\left|\boldsymbol{w}^H \boldsymbol{a}(\theta)\right| \leq \rho(\theta) \quad \forall\theta \in \mathbb{S} \tag{4c}$$

$$|\boldsymbol{w}| \preceq \eta \cdot \boldsymbol{b} \tag{4d}$$

$$\boldsymbol{b} \in \{0, 1\}^N. \tag{4e}$$

Note that in the above formulation (4), $\boldsymbol{b}$ is a binary vector, i.e., its elements can only take the values of zero or one as indicated in (4e). According to (4d), it becomes evident that if an element of $\boldsymbol{b}$ equals zero, the value at the corresponding position in $\boldsymbol{w}$ must be zero as well. Conversely, when an element of $\boldsymbol{b}$ takes one, the corresponding element in $\boldsymbol{w}$ can takes nonzero values. Based on this fact, the problem of minimizing ℓ_0 norm of $\boldsymbol{w}$ can be converted into a problem of minimizing $\mathbf{1}^T\boldsymbol{b}$, as described in (4a). Similar reformulation for sparse problem can be found in [35] and [36].

The formulation (4) is an MIP problem, where some variables (see vector $\boldsymbol{b}$) can only take on integer values, while others (see vector $\boldsymbol{w}$) can take on continuous values. In fact, we can solve problem (4) using MIP solvers, such as GUROBI [33] and CPLEX [34].

C. Sparse Array Synthesis With Discrete Phase Constraints

Phase quantization can reduce system cost and hardware complexity. Under discrete phase constraints, the phase excitation must be chosen from a given discrete alphabet. More specifically, assuming that the quantization bit is D and defining $L = 2^D$, we can obtain the following vector $\boldsymbol{c}$ by enumerating the complex exponentials of the candidate phases

$$\boldsymbol{c} = \left[1, e^{j2\pi/L}, \ldots, e^{j2(L-1)\pi/L}\right]^T. \tag{5}$$

Introducing an $N \times L$ matrix $\mathbf{T}$ to indicate the amplitude excitation, we can further express the weighting vector as

$$\boldsymbol{w} = \mathbf{T}\boldsymbol{c} \tag{6}$$

where the entries of $\mathbf{T}$ are non-negative, i.e., $\mathbf{T} \succeq \mathbf{0}$. Since $\boldsymbol{c}$ is pre-assigned, designing $\boldsymbol{w}$ can be converted to designing for $\mathbf{T}$ with appropriate constraints.

Utilizing the above expression (6) and combining it with the formulation in (4), the problem of sparse array synthesis with discrete phase constraints can be formulated as

$$\min_{\mathbf{T},\mathbf{B}} \quad \mathbf{1}^T\mathbf{B}\mathbf{1} \tag{7a}$$

$$\text{s.t.} \quad \Re\left(\mathbf{c}^H\mathbf{T}^H\boldsymbol{a}\left(\bar{\theta}_0\right)\right) \geq 1 \tag{7b}$$

$$\left|\mathbf{c}^H\mathbf{T}^H\boldsymbol{a}(\theta)\right| \leq \rho(\theta), \theta \in \mathbb{S} \tag{7c}$$

$$\mathbf{T} \succeq \mathbf{0} \tag{7d}$$

$$\mathbf{T} \preceq \eta \cdot \mathbf{B} \tag{7e}$$

$$\mathbf{B}\mathbf{1} \preceq \mathbf{1} \tag{7f}$$

$$\mathbf{B} \in \{0, 1\}^{N\times L} \tag{7g}$$

where (7b) and (7c) constrain the mainlobe and sidelobe level for beampattern. As the discrete phase constraints are considered, we have replaced the equality constraint in (4b) with an inequality constraint in (7b). Similar to the binary vector $\boldsymbol{b}$ in (4), we introduce an $N \times L$ binary matrix $\mathbf{B}$ in (7) to constrain the amplitude matrix $\mathbf{T}$. More specifically, the constraint (7f) ensures that at most one element of each row in matrix $\mathbf{B}$ can be taken as one. With constraint (7f) satisfied, (7e) ensures that at most one element in each row of $\mathbf{T}$ can have a positive value, where η in (7e) plays a similar role as it does in constraint (4d). Finally, to achieve sparse array synthesis, we minimize the number of ones in matrix $\mathbf{B}$ [as described in (7a)]. It should be noted that the formulation (7) is an MIP problem.

D. Two Extensions Under Discrete Phase Constraints

In the preceding section, we formulated the problem of synthesizing a sparse array under discrete phase constraints as an MIP problem. Remarkably, the proposed model can be extended to address more intricate scenarios with various practical constraints. Next, we consider two extensions under discrete phase constraints.

1) Sparse Array Synthesis With Interelement Spacing Constraints: To mitigate the mutual coupling effect between antennas, it is crucial to enforce constraints on the minimum spacing between antennas [17], [18], [19]. According to the formulation (7) and defining $\boldsymbol{z} \triangleq \mathbf{B}\mathbf{1}$, the minimum spacing constraints can be realized by controlling the number of the adjacent ones in the binary vector $\boldsymbol{z}$.

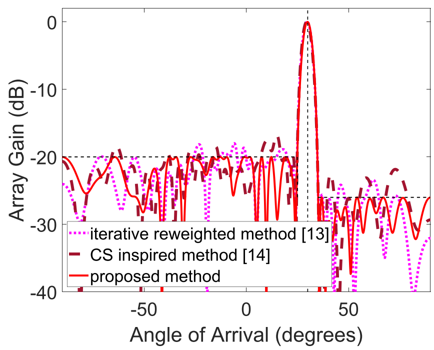


Fig. 1. Comparison of the synthesized patterns.

Specifically, assuming the candidate antennas are uniformly distributed with an interelement spacing d, and the minimum allowable interelement spacing is r, then the sparse array synthesis problem considering minimum spacing constraint can be formulated as

$$\min_{\mathbf{T},\mathbf{B}} \quad \mathbf{1}^T\mathbf{B}\mathbf{1} \tag{8a}$$

$$\text{s.t.} \quad \Re\left(\mathbf{c}^H\mathbf{T}^H\boldsymbol{a}\left(\bar{\theta}_0\right)\right) \geq 1 \tag{8b}$$

$$\left|\mathbf{c}^H\mathbf{T}^H\boldsymbol{a}(\theta)\right| \leq \rho(\theta), \theta \in \mathbb{S} \tag{8c}$$

$$\mathbf{T} \succeq \mathbf{0} \tag{8d}$$

$$\mathbf{T} \preceq \eta \cdot \mathbf{B} \tag{8e}$$

$$\mathbf{B}\mathbf{1} \preceq \mathbf{1} \tag{8f}$$

$$\mathbf{E}\mathbf{B}\mathbf{1} \preceq \mathbf{1} \tag{8g}$$

$$\mathbf{B} \in \{0,1\}^{N\times L} \tag{8h}$$

where the constraint (8g) is newly added comparing to (7). Defining $M \triangleq \lfloor r/d \rfloor + 1$ and $P \triangleq N - M + 1$, we can express the $P \times N$ matrix $\mathbf{E}$ in (8g) as

$$\mathbf{E} \triangleq \begin{pmatrix} \boldsymbol{e}_1^T \\ \boldsymbol{e}_2^T \\ \vdots \\ \boldsymbol{e}_P^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \ddots & & \ddots & & \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

where $\boldsymbol{e}_p$ is an N-dimensional vector, with its values being ones from the pth element to the $(p + M - 1)$th element, and zeros for all other elements, $p = 1, \ldots, P$.

2) Synthesis of Phase-Only Reconfigurable Sparse Arrays: In phase-only reconfigurable arrays, the beam direction or beam shape can be changed solely by adjusting the phase excitation. In this case, we need to jointly design the amplitude excitation, arrangement of antenna position, and (discrete) phase excitation of each beam, according to the given radiation requirements.

Taking the focused-beam scanning as an example and denoting the number of beams as K, we follow (6) and express the weight vector of the kth beam as

$$\boldsymbol{w}_k = \mathbf{T}_k\boldsymbol{c} \quad \forall k \in \mathbb{K} \triangleq \{1, 2, \ldots, K\}. \tag{9}$$

We can then obtain the following formulation:

$$\min_{\{\mathbf{T}_k\}_{k=1}^K, \{\mathbf{B}_k\}_{k=1}^K, \mathbf{t}, \mathbf{s}} \quad \mathbf{1}^T\mathbf{s} \tag{10a}$$

$$\text{s.t.} \quad \Re\left(\mathbf{c}^H\mathbf{T}_k^H\boldsymbol{a}\left(\bar{\theta}_k\right)\right) \geq 1 \quad \forall k \in \mathbb{K} \tag{10b}$$

$$\left|\mathbf{c}^H\mathbf{T}_k^H\boldsymbol{a}(\theta)\right| \leq \rho_k(\theta), \theta \in \mathbb{S}_k \quad \forall k \in \mathbb{K} \tag{10c}$$

$$\mathbf{T}_k\mathbf{1} = \mathbf{t} \quad \forall k \in \mathbb{K} \tag{10d}$$

$$\mathbf{T}_k \succeq \mathbf{0} \quad \forall k \in \mathbb{K} \tag{10e}$$

$$\mathbf{T}_k \preceq \eta \cdot \mathbf{B}_k \quad \forall k \in \mathbb{K} \tag{10f}$$

$$\mathbf{B}_k\mathbf{1} = \mathbf{s} \quad \forall k \in \mathbb{K} \tag{10g}$$

$$\mathbf{s} \preceq \mathbf{1} \tag{10h}$$

$$\mathbf{B}_k \in \{0,1\}^{N\times L} \quad \forall k \in \mathbb{K} \tag{10i}$$

where $\bar{\theta}_k$ represents the mainlobe axis of the kth beam, $\rho_k(\theta)$ is the upper bound level for sidelobe, and $\mathbb{S}_k$ is the sidelobe region. Note that we have introduced K binary matrices (i.e., $\mathbf{B}_k$) in (10). In constraint (10d), a dummy variable $\boldsymbol{t}$ is introduced to ensure that $\mathbf{T}_k\mathbf{1}$ take on the same values for any $k \in \mathbb{K}$. This guarantees that each beam has the same amplitude excitation, i.e., satisfying phase-only control. Constraints (10f)–(10h) ensure that different beams are synthesized by the same array. Finally, the cost function in (10a) aims to minimize the number of antennas, thus achieving sparse array synthesis. Similar to (7) and (8), the formulation (10) is an MIP problem.

III. Numerical Results

In this section, several representative simulations are provided to demonstrate the effectiveness and superiority of the proposed method.[1] The iterative reweighted ℓ_1 norm minimization method in [13], the CS-inspired sparse array synthesis method in [14] and the MMV-FOCUSS method in [15] are tested for comparison if applicable. In all the simulations below, we set η to 5. The number of phase quantization bits is set to 4, i.e., $L = 16$. In our simulations, the proposed MIP models are solved by GUROBI [33]. The simulations are conducted using a computing platform, the processor is Intel Core i7-10750H CPU at 2.60 GHz.

A. Sparse Array Synthesis With Discrete Phase Constraints

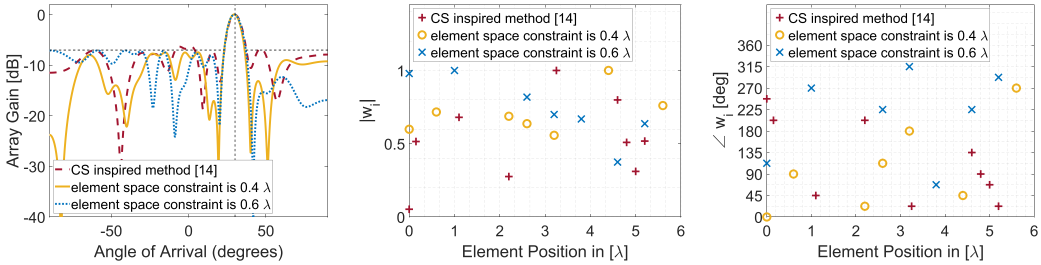
In the first simulation, only discrete phase constraint is considered. In this case, we steer the mainlobe axis to $\bar{\theta}_0 = 30°$. The mainlobe width of the desired focused pattern is 10°, and the nonuniform sidelobe upper level is shown by the black dashed line in Fig. 1. For the iterative reweighted method and the proposed method, we use 80 candidate antennas which are uniformly distributed with an interelement spacing $d = \lambda/3$. For the CS inspired method, we preset 533 candidate antennas with an interelement spacing $d = \lambda/3$.

Fig. 1 shows the resulting beampatterns of three methods. We can see that the radiation pattern of the proposed method meets the expected requirements completely, while the sidelobe levels of the other exceed the desired levels. This is mainly because the reweighted method and the CS inspired method rounds the obtained phases, whereas the proposed method designs discrete phases directly. For the proposed method, the ultimate antenna number is 31 costing 2479 s. The resulting array element numbers for the iterative reweighted method and the CS inspired method are 29 and 34, with computational times of 110 s and 1599 s, respectively.

B. Sparse Array Synthesis With Interelement Spacing Constraints

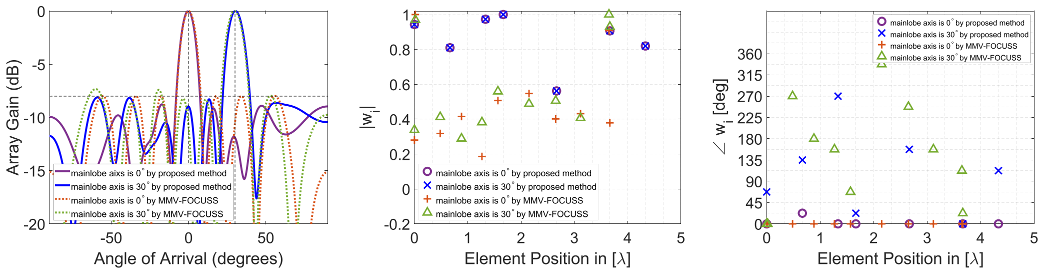
In the second example, we consider interelement spacing constraints in sparse array synthesis. For the proposed method, the number of candidate antennas is 30, and the interelement spacing

[1]The MATLAB codes for the proposed method are available online at https://zhangxuejing7.github.io/HomePage/.



(a) (b) (c)

Fig. 2. Simulation results of sparse array synthesis with two different minimum interelement spacing constraints. (a) Synthesized patterns. (b) Distributions of amplitude excitations for the synthesized sparse array. (c) Distributions of phase excitations for the synthesized sparse array.



(a) (b) (c)

Fig. 3. Simulation results for the synthesis of phase-only reconfigurable sparse arrays. (a) Synthesized patterns considering reconfigurability. (b) Distributions of amplitude excitations for the synthesized sparse array. (c) Distributions of phase excitations for the synthesized sparse array.

is $d = 0.2\lambda$. For CS inspired method, these parameters are 120 and $d = \lambda/20$, respectively. We consider two scenarios with minimum interelement spacing of 0.4λ and 0.6λ, respectively. In both cases, we steer the beam axis to $\bar{\theta}_0 = 30^\circ$ and adopt a low sidelobe focused radiation pattern as the desired one.

For the proposed method, the resulting antenna number of the synthesized array is 7 in both scenarios. The ultimate antenna number for the CS-inspired method is 9. Fig. 2 shows the simulation results of two methods. From the synthesized patterns as depicted in Fig. 2(a), one can see that the proposed method can synthesize qualified radiation patterns. Fig. 2(b) and (c) provides the distributions of amplitude and phase excitations with the arrangement of antenna positions. For the proposed method, we can see from Fig. 2(c) that all the phase excitations satisfy the preset discrete phase constraints, and the spacing between antennas satisfies the minimum spacing constraint. In contrast, for the CS-inspired method, the minimum spacing constraint is not satisfied. Also, it should be noted that the running time of the proposed method (5168 s) is longer than that of the CS-inspired method (2571 s).

C. Synthesis of Phase-Only Reconfigurable Sparse Arrays

For the last simulation, we implement the synthesis of phase-only reconfigurable sparse arrays. For simplicity, we consider a two-beam scenario. To synthesize a reconfigurable sparse array radiating low sidelobe focused patterns, we steer the mainlobe axes to $\bar{\theta}_1 = 0^\circ$ and $\bar{\theta}_2 = 30^\circ$, respectively. For the proposed method, we use a candidate antenna array consists of 30 antennas with inter element spacing $\lambda/3$. For comparison, we conduct MMV-FOCUSS method using 600 candidate antennas and inter-element spacing $\lambda/60$.

Fig. 3 presents the simulation results with the above parameters. We can see from Fig. 3(a) that the two reconfigurable radiation patterns of the proposed method (with 7 antennas) satisfy the given requirements. However, the patterns of the MMV-FOCUSS method (with 11 antennas) exceed the expected sidelobe due to phase quantization. The distributions of amplitude and phase excitations with the arrangement of antenna positions are presented in Fig. 3(b) and (c), respectively. Notice that after considering reconfigurability, the two sets of amplitude excitations of the proposed method are the same, thus satisfying the requirement of phase-only control. In addition, the proposed method exhibits two distinct sets of phase distributions, both of which satisfy the discrete phase constraints. Finally, it can be observed from Fig. 3(b) that the amplitude excitations of the two beams are different for the MMV-FOCUSS method. Therefore, the MMV-FOCUSS method is unable to synthesize phase-only reconfigurable sparse array in this scenario. In this simulation, the MMV-FOCUSS method can be completed within a few seconds, while the running time of the proposed method exceeds 6000 s.

IV. Conclusion

In this letter, we have presented a novel approach for synthesizing sparse arrays with discrete phase constraints via MIP. Our proposed approach optimizes both antenna positions and amplitude/phase excitations subject to the given discrete phase constraints. Our approach can handle various practical constraints, such as minimum spacing constraints between antennas, phase-only constraints in reconfigurable arrays. For different scenarios, we have formulated different MIP models that can be solved using off-the-shelf solver. We have presented representative simulations to demonstrate the effectiveness of the proposed method. Although exhibiting superior performance in sparse array synthesis, the proposed method has the drawback of long computational time. Future work will incorporate array perturbations into consideration to improve the robustness of sparse array synthesis.

REFERENCES

[1] D. Dai, M. Yao, H. Ma, W. Jin, and F. Zhang, "An asymmetric mapping method for the synthesis of sparse planar arrays," *IEEE Antennas Wireless Propag. Lett.*, vol. 17, no. 1, pp. 70–73, Jan. 2018.

[2] S. K. Goudos, K. Siakavara, T. Samaras, E. E. Vafiadis, and J. N. Sahalos, "Sparse linear array synthesis with multiple constraints using differential evolution with strategy adaptation," *IEEE Antennas Wireless Propag. Lett.*, vol. 10, pp. 670–673, 2011.

[3] L. F. Yepes, D. H. Covarrubias, M. A. Alonso, and R. Ferrús, "Hybrid sparse linear array synthesis applied to phased antenna arrays," *IEEE Antennas Wireless Propag. Lett.*, vol. 13, pp. 185–188, 2014.

[4] K. Yan and Y. Lu, "Sidelobe reduction in array-pattern synthesis using genetic algorithm," *IEEE Trans. Antennas Propag.*, vol. 45, no. 7, pp. 1117–1122, Jul. 1997.

[5] D. W. Boeringer and D. H. Werner, "Particle swarm optimization versus genetic algorithms for phased array synthesis," *IEEE Trans. Antennas Propag.*, vol. 52, no. 3, pp. 771–779, Mar. 2004.

[6] V. Murino, A. Trucco, and C. S. Regazzoni, "Synthesis of unequally spaced arrays by simulated annealing," *IEEE Trans. Signal Process.*, vol. 44, no. 1, pp. 119–122, Jan. 1996.

[7] Y. Liu, Z. Nie, and Q. H. Liu, "Reducing the number of elements in a linear antenna array by the matrix pencil method," *IEEE Trans. Antennas Propag.*, vol. 56, no. 9, pp. 2955–2962, Sep. 2008.

[8] P. Gu, Z. He, J. Xu, K. W. Leung, and R. S. Chen, "Design of wide scanning sparse planar array using both matrix-pencil and space-mapping methods," *IEEE Antennas Wireless Propag. Lett.*, vol. 20, no. 2, pp. 140–144, Feb. 2021.

[9] H. Shen, B. Wang, and X. Li, "Shaped-beam pattern synthesis of sparse linear arrays using the unitary matrix pencil method," *IEEE Antennas Wireless Propag. Lett.*, vol. 16, pp. 1098–1101, 2017.

[10] C. Zhu, X. Li, Y. Liu, L. Liu, and Q. H. Liu, "An extended generalized matrix pencil method to synthesize multiple-pattern frequency-invariant linear arrays," *IEEE Antennas Wireless Propag. Lett.*, vol. 16, pp. 2311–2315, 2017.

[11] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[12] B. Fuchs, "Application of convex relaxation to array synthesis problems," *IEEE Trans. Antennas Propag.*, vol. 62, no. 2, pp. 634–640, Feb. 2014.

[13] B. Fuchs, "Synthesis of sparse arrays with focused or shaped beampattern via sequential convex optimizations," *IEEE Trans. Antennas Propag.*, vol. 60, no. 7, pp. 3499–3503, Jul. 2012.

[14] D. Pinchera, M. D. Migliore, F. Schettino, M. Lucido, and G. Panariello, "An effective compressed-sensing inspired deterministic algorithm for sparse array synthesis," *IEEE Trans. Antennas Propag.*, vol. 66, no. 1, pp. 149–159, Jan. 2018.

[15] F. Yan, P. Yang, F. Yang, L. Zhou, and M. Gao, "Synthesis of pattern reconfigurable sparse arrays with multiple measurement vectors focuss method," *IEEE Trans. Antennas Propag.*, vol. 65, no. 2, pp. 602–611, Feb. 2017.

[16] S. E. Nai, W. Ser, Z. L. Yu, and H. Chen, "Beampattern synthesis for linear and planar arrays with antenna selection by convex optimization," *IEEE Trans. Antennas Propag.*, vol. 58, no. 12, pp. 3923–3930, Dec. 2010.

[17] C. Yan, P. Yang, Z. Xing, and S. Y. Huang, "Synthesis of planar sparse arrays with minimum spacing constraint," *IEEE Antennas Wireless Propag. Lett.*, vol. 17, no. 6, pp. 1095–1098, Jun. 2018.

[18] P. You, Y. Liu, S. L. Chen, K. D. Xu, W. Li, and Q. H. Liu, "Synthesis of unequally spaced linear antenna arrays with minimum element spacing constraint by alternating convex optimization," *IEEE Antennas Wireless Propag. Lett.*, vol. 16, pp. 3126–3130, 2017.

[19] M. D. Urso, G. Prisco, and R. M. Tumolo, "Maximally sparse, steerable, and nonsuperdirective array antennas via convex optimizations," *IEEE Trans. Antennas Propag.*, vol. 64, no. 9, pp. 3840–3849, Sep. 2016.

[20] F. Yang, S. Yang, Y. Chen, S. Qu, and J. Hu, "Synthesis of sparse antenna arrays subject to constraint on directivity via iterative convex optimization," *IEEE Antennas Wireless Propag. Lett.*, vol. 20, no. 8, pp. 1498–1502, Aug. 2021.

[21] Y. Lee, "Adaptive beamforming with continuous/discrete phase shifters via convex relaxation," *IEEE Open J. Antennas Propag.*, vol. 3, pp. 557–567, 2022.

[22] C. Baird and G. Rassweiler, "Adaptive sidelobe nulling using digitally controlled phase-shifters," *IEEE Trans. Antennas Propag.*, vol. AP-24, no. 5, pp. 638–647, Sep. 1976.

[23] W. P. M. N. Keizer, "Low sidelobe phased array pattern synthesis with compensation for errors due to quantized tapering," *IEEE Trans. Antennas Propag.*, vol. 59, no. 12, pp. 4520–4524, Dec. 2011.

[24] S. K. Goudos, "Antenna design using binary differential evolution: Application to discrete-valued design problems," *IEEE Antennas Propag. Mag.*, vol. 59, no. 1, pp. 74–93, Feb. 2017.

[25] T. H. Ismail and Z. M. Hamici, "Array pattern synthesis using digital phase control by quantized particle swarm optimization," *IEEE Trans. Antennas Propag.*, vol. 58, no. 6, pp. 2142–2145, Jun. 2010.

[26] S. Bourguignon, J. Ninin, H. Carfantan, and M. Mongeau, "Exact sparse approximation problems via mixed-integer programming: Formulations and computational performance," *IEEE Trans. Signal Process.*, vol. 64, no. 6, pp. 1405–1419, Mar. 2016.

[27] J. Ignacio Echeveste, M. Á. González de Aza, J. Rubio, and J. Zapata, "Synthesis of coupled antenna arrays using digital phase control via integer programming," *Microw., Antennas Propag.*, vol. 12, no. 6, pp. 999–1003, 2018.

[28] K. Miao, Y. Zhang, C. Yao, and H. Sun, "Sidelobe suppression by phase-only tapering with integer linear programming," in *Proc. Int. Conf. Radar*, 2021, pp. 1989–1992.

[29] O. M. Bucci, G. Mazzarella, and G. Panariello, "Reconfigurable arrays by phase-only control," *IEEE Trans. Antennas Propag.*, vol. 39, no. 7, pp. 919–925, Jul. 1991.

[30] F. Dobias and J. Gunther, "Reconfigurable array antennas with phase only control of quantized phase shifters," in *Proc. IEEE 45th Veh. Technol. Conf.*, 1995, pp. 35–39.

[31] A. F. Morabito, A. Massa, P. Rocca, and T. Isernia, "An effective approach to the synthesis of phase-only reconfigurable linear arrays," *IEEE Trans. Antennas Propag.*, vol. 60, no. 8, pp. 3622–3631, Aug. 2012.

[32] R. Vescovo, "Reconfigurability and beam scanning with phase-only control for antenna arrays," *IEEE Trans. Antennas Propag.*, vol. 56, no. 6, pp. 1555–1565, Jun. 2008.

[33] *Gurobi Optimizer Reference Manual*, Gurobi Optimization, Briarwood, TX, USA, 2023.

[34] *IBM ILOG CPLEX Optimization Studio CPLEX Users Manual*, IBM, Raleigh, NC, USA, 2018.

[35] S. Jokar and M. Pfetsch, "Exact and approximate sparse solutions of underdetermined linear equations," *SIAM J. Sci. Comput.*, vol. 31, no. 1, pp. 23–44, 2008.

[36] N. B. Karahanoglu, H. Erdogan, and S. I. Birbil, "A mixed integer linear programming formulation for the sparse recovery problem in compressed sensing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 2013, pp. 5870–5874.