Pattern Synthesis for Arbitrary Arrays via Weight Vector Orthogonal Decomposition

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***Abstract*—This paper presents a new scheme based on Weight vector ORthogonal Decomposition (WORD) to control the array response at a given direction and a novel WORD-based approach to pattern synthesis for arbitrary arrays. The central concept of the proposed methods stems from the adaptive array theory. More precisely, it is found that the inverse of the noise-plus-interference covariance matrix in adaptive beamforming can be regarded as a linear combination of two orthogonal projection matrices, and, accordingly, the optimal weight vector is a linear combination of two orthogonal vectors. With such an observation, the WORD scheme is developed to design the desired weight vector. It is shown that the array response at a given direction can be precisely adjusted to an arbitrary level, by simply determining appropriate combination coefficients for those two orthogonal vectors. Furthermore, a closed-form expression of the weight vector can be achieved by introducing a new cost function that measures pattern variation. By employing the WORD scheme successively, a novel approach to pattern synthesis for arbitrary arrays is devised. At each implementation step of this approach, the array pattern is adjusted in a point-by-point manner by successively modifying the weight vector. As such, both the sidelobe and mainlobe regions can be flexibly synthesized. Numerical examples are provided to demonstrate the effectiveness and flexibility of the WORD scheme in array response control at a single direction as well as pattern synthesis.**

***Index Terms*—Array pattern synthesis, array response control, adaptive array theory, array signal processing.**

I. Introduction

ARRAY antenna has found numerous applications to radar, navigation, wireless communication and other fields [1]. Methods for optimal array antenna design play a critical role in, for example, improving system performance and reducing cost. In particular, determining the complex weights for array elements to achieve a desired beampattern, i.e., pattern synthesis, is a fundamental problem [2]–[9]. With regard to this problem, it is expected to control the sidelobe of array response to achieve a pencil beam pattern or to realize a shaped beam pattern complying to a given mask. For instance, in radar systems, it is desirable to mitigate returns from interfering signals by designing a pattern with several nulls at specified directions. In some communication systems, it may be required to shape multiple-beams patterns for multi-user reception. Additionally, synthesizing a pattern with broad mainlobe is helpful for extending monitoring areas in satellite remote sensing.

In the past few decades, pattern synthesis has been attracted much research interest and numerous techniques have been developed. The classical algorithms [10]–[12] have closed-form expressions, however, they are limited to some specific array geometries or array patterns. Many global optimization based methods design array patterns by finding the optimal solutions via stochastic approaches such as genetic algorithm (GA) [13], particle swarm optimization (PSO) method [14] and simulated annealing (SA) method [15]. In general, methods of this kind are time-consuming. Another stream of pattern synthesis approaches [16]–[19] has been devised with the help of adaptive array theory [20]–[22]. These methods iteratively minimize the deviation between the synthesized and desired patterns. It is noteworthy that some key parameters such as the powers of virtual interferences are selected in an *ad hoc* way [17]–[19], and deterministic schemes on parameter selection need investigation.

More recently, pattern nulling techniques based on innovative architectural solutions requiring the ability of connecting or disconnecting the array elements have been reported in [23] and [24]. For these approaches, radio-frequency (RF) switches in the beamforming network have been used for reconfiguring the antenna pattern. In [25], the almost difference sets (ADSs) is considered for the design of thinned planar arrays, where the peak sidelobe levels (PSLs) can be predictable and deducible from the knowledge of the array aperture. The synthesis of modular and contiguously-clustered arrays has been addressed in [26] through an innovative sparsity-regularized methodology. The issue of maximum efficiency beam synthesis is considered and further solved in [27] with the aid of generalized eigenvalue decomposition. It should be noted that the above methods in [23]–[27] focus on linear or planar arrays, their extensions to other array geometries need further discussions and more efforts.

A different class of algorithms for the array pattern synthesis problem have been devised along with recent advances in

Manuscript received June 8, 2017; revised October 31, 2017; accepted December 19, 2017. Date of publication December 25, 2017; date of current version December 26, 2018. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Frederic Pascal. This work was supported in part by National Nature Science Foundation of China under Grant 61671139, Grant 61771316, and Grant 61701499, in part by the Foundation of the Department of Education of Guangdong Province under Grant 2016KTSCX125, and in part by the Nature Science Foundation of Guangdong Province 2017A030313341. *(Corresponding author: Xuejing Zhang.)*

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Digital Object Identifier 10.1109/TSP.2017.2787143

convex optimization [28]. For instance, in [29], Lebret and Boyd proposed to express pattern synthesis problems as convex optimization problems, which can be solved using interior-point methods. However, as emphasized therein, this approach is limited to some simple arrays where design problems are convex. For linear and planar arrays with non-convex lower bound constraint on the beampattern, conjugate symmetric beamforming weights have been utilized in [30] to reformulate the problem as a convex one. In [31], Fuchs introduced the idea of semidefinite relaxation (SDR) [32] to the problem of pattern synthesis, and succeeded to synthesize patterns with upper and lower constraints. Note that the resultant pattern may not meet the original design requirements, since the relaxation can only lead to an approximate solution. Other convex optimization based methods can be found in [33]–[35], where optimization toolbox, e.g., [36], is usually adopted to solve the resultant optimization problems. Apart from the aforementioned methods, there also exist a few approaches attempting to control or synthesize patterns by utilizing the least-squares method [37], [38], employing the Fast Fourier Transformation (FFT) [39] or excitation matching approach [40]. The interested reader is referred to the recent work [41] for a more comprehensive literature review of array response control and pattern synthesis.

Generally speaking, the approaches mentioned above cannot flexibly control array response pattern. For instance, the pattern has to be completely re-synthesized even if only some minor changes of the pattern need to be made. The recently developed accurate array response control (A^{2}RC) algorithm [42] gives a closed-form expression of weight vector to precisely control the response level at a preassigned direction. Nevertheless, the key parameter (i.e., μ_{opt} as denoted in [42]) is complex-valued, which has to be determined by solving an optimization problem with computationally inefficient global search or in an empirical manner. More importantly, as shall be shown later, the ultimate weight vector of A^{2}RC may lead to severe pattern distortion due to empirical selection of the parameter μ. This motivates us to propose a new way to flexibly and precisely control a response pattern (or design a desired weight vector) on the basis of a given pattern (or a given weight vector). For this purpose, an elaborate analysis of the optimal weight vector in adaptive beamforming is firstly given. It is shown that the optimal weight vector, which maximizes the signal-to-interference-plus-noise ratio (SINR) to suppress a single interference, can be equivalently expressed as a linear combination of two orthogonal vectors with specific weighting coefficients. This implies that any response level at a single direction can be achieved via weight vector orthogonal decomposition (WORD) and weighting coefficients selection.

With this new WORD scheme, it is further proposed in this paper to update the weight vector from its initial value in a step-by-step manner to synthesize a desired pattern. In each step, the response at one selected angle is adjusted to its desired level under the condition that the pattern distortion at other regions is minimized. On this basis, a new criterion is established to measure pattern distortion and further help to select the optimal parameter in a simple way. Comparing with conventional approaches, the proposed WORD-based pattern synthesis method works in a deterministic manner and is able to flexibly and accurately adjust the array responses without leading to a pattern distortion. Unlike the A^{2}RC approach [42], the key parameter (i.e., β) is real-valued and can be simply determined from a testing problem. Furthermore, the problem of pattern distortion can be readily avoid in the proposed approach. Additionally, the proposed pattern synthesis method can also be applied to arbitrary arrays including commonly used uniform linear arrays (ULAs), nonisotropic random arrays and two-dimensional arrays. Note that although the WORD methodology stems from the adaptive array theory, as can be seen later, the resulting pattern synthesis problem is subject to certain constraints.

The paper is organized as follows. In Section II, the problem formulation of pattern synthesis and adaptive array theory are briefly given. The WORD scheme is devised in Section III and its application to pattern synthesis is discussed in Section IV. In Section V, comparisons with previous A^{2}RC method and the proposed one is elaborated. Numerical examples are conducted in Section VI to demonstrate the performance of the proposed method. Conclusions are drawn in Section VII.

II. Preliminaries

We now give some preliminaries of adaptive array theory, which will be exploited for pattern synthesis with constraints. Let us consider an N-element antenna array with arbitrary geometry. In what follows, we consider the one-dimensional case for the sake of notational simplicity. Nevertheless, the proposed methods herein can be straightforwardly applied to two-dimensional scenarios. The steering vector associated with direction θ is given by

$$\mathbf{a}(\theta) = \left[g_1(\theta)e^{j\phi_1(\theta)}, \ldots, g_N(\theta)e^{j\phi_N(\theta)}\right]^{\mathrm{T}} \tag{1}$$

where $(\cdot)^{\mathrm{T}}$ is the transpose operator, $j = \sqrt{-1}$ is the imaginary unit, $g_n(\theta)$ represents the radiation pattern of the nth element (we have $g_n(\theta) = 1$ when the antenna is isotropic), $\phi_n(\theta) = \omega\tau_n(\theta)$ stands for the phase delay of the nth element, where ω is the operation frequency, $\tau_n(\theta)$ represents the time-delay between the nth element and reference point. The far-field array response (array factor) can thus be written as

$$f(\theta) = \sum_{n=1}^{N} w_n^* g_n(\theta)e^{j\phi_n(\theta)} = \mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta) \tag{2}$$

where $\mathbf{w} = [w_1, w_2, \ldots, w_N]^{\mathrm{T}}$ is the array weight vector, $(\cdot)^*$ and $(\cdot)^{\mathrm{H}}$ denote the conjugate operator and conjugate transpose operator, respectively.

The problem of pattern synthesis amounts to find an appropriate weight vector $\mathbf{w}$ such that the amplitude of the array response, i.e., $|f(\theta)|$, meets certain requirements. Numerous techniques have been proposed in the literature [2]–[19], among which various algorithms [16]–[19] are based on adaptive array theory. In brief, in adaptive array processing, assume that θ_0 is the direction of arrival of the desired signal, to suppress the interferences and noise, the optimal weight vector which maximizes the SINR is given by

$$\mathbf{w}_\star = \mathbf{R}_{n+i}^{-1}\,\mathbf{a}(\theta_0) \tag{3}$$

where $\mathbf{R}_{n+i}$ is the noise-plus-interference covariance matrix. For instance, in the scenario of a single interference and noise being white, the noise-plus-interference covariance matrix can be described as

$$\mathbf{R}_{n+i} = \sigma_n^2 \mathbf{I} + \sigma_i^2 \mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i) \tag{4}$$

where σ_n^2 and σ_i^2 stand for noise power and interference power, respectively, $\mathbf{a}(\theta_i)$ is the interference steering vector, $\mathbf{I}$ is the identity matrix with proper dimension. It has been shown in [17]–[19] that the array pattern can be synthesized by selecting the powers of the interferences in an *ad hoc* way. In the following sections, a new parameterization of the optimal weight vector (3) is introduced and a deterministic scheme for parameter selection is devised.

III. Array Response Control via WORD

A. New Parameterization of the Optimal Weight Vector

To begin with, let us denote by $\mathcal{R}(\mathbf{a}(\theta_i))$ the column space of $\mathbf{a}(\theta_i)$, then the projection matrix onto $\mathcal{R}(\mathbf{a}(\theta_i))$ can be expressed as

$$\mathbf{P}_{[\mathbf{a}(\theta_i)]} = \mathbf{a}(\theta_i)(\mathbf{a}^{\mathrm{H}}(\theta_i)\mathbf{a}(\theta_i))^{-1}\mathbf{a}^{\mathrm{H}}(\theta_i) = \frac{\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i)}{\|\mathbf{a}(\theta_i)\|_2^2} \tag{5}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Accordingly, the projection matrix onto the orthogonal complement of $\mathcal{R}(\mathbf{v})$ (i.e., $\mathcal{R}^{\perp}(\mathbf{v})$) is thus given by

$$\mathbf{P}^{\perp}_{[\mathbf{a}(\theta_i)]} = \mathbf{I} - \mathbf{P}_{[\mathbf{a}(\theta_i)]}. \tag{6}$$

Now, considering $\mathbf{R}_{n+i}$ in (4), an equivalent parameterization of $\mathbf{R}^{-1}_{n+i}$ can be obtained by applying the Woodbury lemma [45] as follows

$$\begin{aligned}\mathbf{R}^{-1}_{n+i} &= \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\sigma_i^2\|\mathbf{a}(\theta_i)\|_2^2}{\sigma_n^2 + \sigma_i^2\|\mathbf{a}(\theta_i)\|_2^2}\cdot\frac{\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i)}{\|\mathbf{a}(\theta_i)\|_2^2}\right)\\ &= \frac{1}{\sigma_n^2}\left(\mathbf{I} - \mathbf{P}_{[\mathbf{a}(\theta_i)]} + \frac{\sigma_n^2}{\sigma_n^2 + \sigma_i^2\|\mathbf{a}(\theta_i)\|_2^2}\mathbf{P}_{[\mathbf{a}(\theta_i)]}\right)\\ &= \frac{1}{\sigma_n^2}\left(\mathbf{P}^{\perp}_{[\mathbf{a}(\theta_i)]} + \beta\mathbf{P}_{[\mathbf{a}(\theta_i)]}\right)\end{aligned} \tag{7}$$

where β is a real number associated with σ_n^2, σ_i^2 and $\mathbf{a}(\theta_i)$ as

$$\beta = \frac{\sigma_n^2}{\sigma_n^2 + \sigma_i^2\|\mathbf{a}(\theta_i)\|_2^2}. \tag{8}$$

It can be noticed from (7) that $\mathbf{R}^{-1}_{n+i}$ is a linear combination of $\mathbf{P}^{\perp}_{[\mathbf{a}(\theta_i)]}$ and $\mathbf{P}_{[\mathbf{a}(\theta_i)]}$ with specific coefficients. Consequently, the optimal weight vector in (3) can be parameterized as

$$\begin{aligned}\mathbf{w}_{\star} &= \left(\mathbf{P}^{\perp}_{[\mathbf{a}(\theta_i)]} + \beta\mathbf{P}_{[\mathbf{a}(\theta_i)]}\right)\mathbf{a}(\theta_0) = \mathbf{w}_{(0)\perp} + \beta\mathbf{w}_{(0)\|}\\ &= \left[\,\mathbf{w}_{(0)\perp}\ \mathbf{w}_{(0)\|}\,\right]\left[\,1\ \beta\,\right]^{\mathrm{T}}\end{aligned} \tag{9}$$

where the common factor $1/\sigma_n^2$ is omitted for brevity, since it does not affect the performance of the beamformer and the shape of the beampattern. In the above expression (9), $\mathbf{w}_{(0)\perp}$

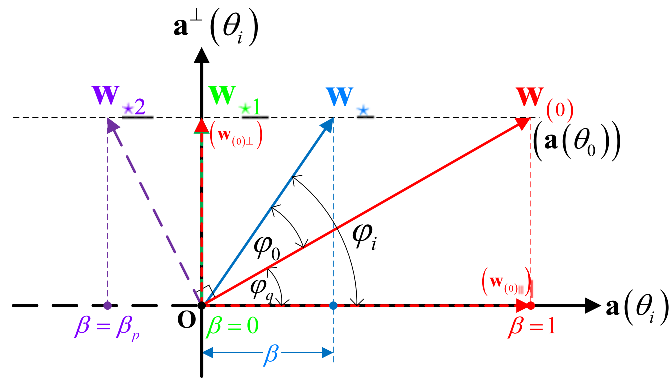

Fig. 1. Illustration of the orthogonal decomposition of $\mathbf{w}_{\star}$.

and $\mathbf{w}_{(0)\|}$ are given by

$$\mathbf{w}_{(0)\perp} = \mathbf{P}^{\perp}_{[\mathbf{a}(\theta_i)]}\mathbf{w}_{(0)} \tag{10a}$$

$$\mathbf{w}_{(0)\|} = \mathbf{P}_{[\mathbf{a}(\theta_i)]}\mathbf{w}_{(0)} \tag{10b}$$

where $\mathbf{w}_{(0)}$ denotes the quiescent weight vector as

$$\mathbf{w}_{(0)} = \mathbf{a}(\theta_0) \tag{11}$$

which actually corresponds to the optimal beamformer in the presence of white noise only, i.e., when $\sigma_i^2 = 0$ or $\beta = 1$.

The parameterization in (9) provides an equivalent expression of the optimal weight vector and illustrates that $\mathbf{w}_{\star}$ is a linear combination of two orthogonal vectors $\mathbf{w}_{(0)\perp}$ and $\mathbf{w}_{(0)\|}$ with coefficients 1 and β, respectively. It can be seen that β is the key parameter to shape a notch at θ_i so as to suppress the interference. The specific value of β in (8) contributes to an optimal weight vector which results in a certain response level (depth of the notch) at θ_i. In particular, we shall show that β can be non-negative for response control, although in adaptive processing it is restricted to this condition since the interference power is non-negative.

B. Relationship Between Array Response and β

In order to explore the issue of how does β in (9) affect the array response level at the given direction θ_i, let us first define the angle $\varphi(\mathbf{u},\mathbf{v})$ between two complex vectors $\mathbf{u}$ and $\mathbf{v}$ as $\cos(\varphi(\mathbf{u},\mathbf{v})) \triangleq \frac{|\mathbf{u}^{\mathrm{H}}\mathbf{v}|}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2}$, where $\varphi(\mathbf{u},\mathbf{v}) \in [0, \pi/2]$. Moreover, we define the normalized power response associated with a weight vector $\mathbf{w}$ as

$$L(\theta,\theta_0) \triangleq \frac{|\mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta)|^2}{|\mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta_0)|^2}. \tag{12}$$

Hence, substituting $\mathbf{w}_{\star}$ in (9) into $L(\theta_i,\theta_0)$, one gets

$$L_{\star}(\theta_i,\theta_0) = \frac{\|\mathbf{a}(\theta_i)\|_2^2\cdot\cos^2(\varphi(\mathbf{w}_{\star},\mathbf{a}(\theta_i)))}{\|\mathbf{a}(\theta_0)\|_2^2\cdot\cos^2(\varphi(\mathbf{w}_{\star},\mathbf{a}(\theta_0)))}. \tag{13}$$

It can be observed that the response at θ_i depends on the angles between $\mathbf{w}_{\star}$ and $\mathbf{a}(\theta_i)$ as well as $\mathbf{w}_{\star}$ and $\mathbf{a}(\theta_0)$. Furthermore, according to (9), it is known that $L_{\star}(\theta_i,\theta_0)$ is a function of β which is to be determined.

To have a geometrical perspective, $\mathbf{w}_{\star}$ and its components are depicted in Fig. 1, where $\varphi_i \triangleq \varphi(\mathbf{w}_{\star},\mathbf{a}(\theta_i)) =$

TABLE I
RESPONSES FOR DIFFERENT SELECTIONS OF β

β	φ_0	φ_i	$L_\star(\theta_i,\theta_0)$
1	0	φ_q	L_q
0	$\mathrm{atan}\,(-1/\beta_p)$	$\pi/2$	0
β_p	$\pi/2$	$\pi/2-\varphi_q$	$+\infty$
$\pm\infty$	φ_q	0	L_∞

$\varphi(\mathbf{w}_{(0)\perp}+\beta\mathbf{w}_{(0)\|},\mathbf{a}(\theta_i))$ and $\varphi_0 \triangleq \varphi(\mathbf{w}_\star,\mathbf{a}(\theta_0)) = \varphi(\mathbf{w}_{(0)\perp} + \beta\mathbf{w}_{(0)\|},\mathbf{a}(\theta_0))$. It is seen that $\mathbf{w}_{(0)\|}$ is the component obtained by projecting $\mathbf{w}_{(0)}$ along $\mathbf{a}(\theta_i)$, while $\mathbf{w}_{(0)\perp}$ is the component that projects $\mathbf{w}_{(0)}$ into $\mathcal{R}^\perp(\mathbf{a}(\theta_i))$. Moreover, φ_i and φ_0 change along with β. For instance, if $\beta=1$, we have $\varphi_0=0$ and $\varphi_i=\varphi_q$ where φ_q denotes the angle between $\mathbf{w}_0$ and $\mathbf{a}(\theta_i)$, i.e., $\varphi_q \triangleq \varphi(\mathbf{w}_0,\mathbf{a}(\theta_i))$. In this case, we have $\mathbf{w}_\star=\mathbf{w}_0$ and hence

$$L_\star(\theta_i,\theta_0)\big|_{\beta=1} = \frac{|\mathbf{w}_0^{\mathrm{H}}\mathbf{a}(\theta_i)|^2}{|\mathbf{w}_0^{\mathrm{H}}\mathbf{a}(\theta_0)|^2} \triangleq L_q \tag{14}$$

where L_q is the normalized quiescent power response at θ_i.

When attempting to decrease β from 1, it is seen that both φ_i and φ_0 begin to increase. In particular, when $\beta=0$, φ_i achieves its maximum value $\pi/2$, and the corresponding optimal weight vector, i.e., $\mathbf{w}_{\star 1}$ in Fig. 1, becomes perpendicular to $\mathbf{a}(\theta_i)$, so we have

$$L_\star(\theta_i,\theta_0)\big|_{\beta=0} = 0. \tag{15}$$

In other words, a deep notch can be obtained at θ_i by setting β as zero.

We now proceed to examining what $L_\star(\theta_i,\theta_0)$ will be if $\beta<0$. Fig. 1 depicts a special case where the optimal weight vector $\mathbf{w}_{\star 2}$ is orthogonal to $\mathbf{a}(\theta_0)$. In this case, we have $\beta=\beta_p<0$, $\frac{|\beta_p|\cdot\|\mathbf{w}_{(0)\|}\|_2}{\|\mathbf{w}_{(0)\perp}\|_2} = \frac{\|\mathbf{w}_{(0)\perp}\|_2}{\|\mathbf{w}_{(0)\|}\|_2}$ and $\beta_p = -\frac{\|\mathbf{w}_{(0)\perp}\|_2^2}{\|\mathbf{w}_{(0)\|}\|_2^2}$. It can be verified that the associated weight vector is orthogonal to $\mathbf{a}(\theta_0)$, i.e., $\mathbf{w}_\star^{\mathrm{H}}\mathbf{a}(\theta_0) = (\mathbf{w}_{(0)\perp}+\beta_p\mathbf{w}_{(0)\|})^{\mathrm{H}}(\mathbf{w}_{(0)\perp}+\mathbf{w}_{(0)\|})=0$. As a result, we have

$$L_\star(\theta_i,\theta_0)\big|_{\beta=\beta_p} = +\infty. \tag{16}$$

Besides the above situations, there are two more cases, i.e., β tends to be $+\infty$ and $-\infty$, being worthy of consideration. From Fig. 1, it is readily found that $\beta\to+\infty$ and $\beta\to-\infty$ corresponds to case that $\varphi_i=0$ and $\varphi_0=\varphi_q$. Substitute $\varphi_i=0$ and $\varphi_0=\varphi_q$ into (13), we have

$$L_\star(\theta_i,\theta_0)\big|_{\beta\to\pm\infty} = \frac{\|\mathbf{a}(\theta_i)\|_2^2}{L_q\|\mathbf{a}(\theta_0)\|_2^2} \triangleq L_\infty. \tag{17}$$

The aforementioned typical cases are summarized in Table I.

In order to further investigate the behavior of $L_\star(\theta_i,\theta_0)$ with respect to β, the monotonicity of $L_\star(\theta_i,\theta_0)$ is considered. As shown in Appendix A, in the general case of $\mathbf{a}^{\mathrm{H}}(\theta_0)\mathbf{a}(\theta_i)\neq 0$, we have

$$\frac{\partial L_\star(\theta_i,\theta_0)}{\partial\beta}\begin{cases} >0, & \text{for } \beta\in(-\infty,\beta_p)\cup(0,+\infty) \\ <0, & \text{for } \beta\in(\beta_p,0) \end{cases}. \tag{18}$$

This indicates that $L_\star(\theta_i,\theta_0)$ is monotonically non-decreasing in $(-\infty,\beta_p)$ or $(0,+\infty)$, whereas monotonically non-increasing in $(\beta_p,0)$. As discussed earlier, the normalized response level tends to be infinity if $\beta=\beta_p$ and drops to be zero if $\beta=0$. This implies that any non-negative array response level at θ_i can be achieved by varying β in $\mathbb{R}$.

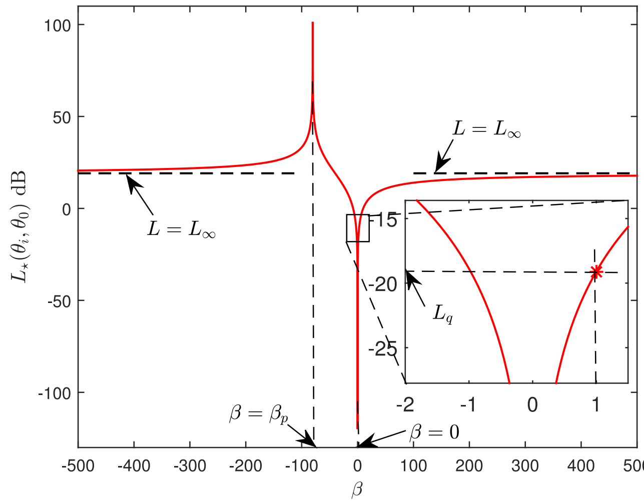

Fig. 2. Curve of $L_\star(\theta,\theta_0)$ versus β for a ULA.

To validate the above results, let us take a ULA of $N=10$ isotropic elements spaced by half wavelength for illustration. The curve of $L_\star(\theta_i,\theta_0)$ versus β for $\theta_0=0^\circ$ and $\theta_i=45^\circ$ is plotted in Fig. 2. In this case, we can figure out that $L_q=-19.1$ dB, $L_\infty=19.1$ dB and $\beta_p=-80.3$. From Fig. 2, we can clearly see that $L_\star(\theta_i,\theta_0)$ becomes infinitely close to L_∞ as β tends to infinity. When $\beta=\beta_p$ is taken, a peak is formed and the response level at θ_i turns to be infinite. Moreover, Fig. 2 shows that $L_\star(\theta_i,\theta_0)$ attains its minimum when $\beta=0$ and is exactly equal to L_q when $\beta=1$. In addition, the monotonicity of $L_\star(\theta_i,\theta_0)$ is clearly displayed in Fig. 2.

It should be pointed out that the above results are obtained based on fact that the previous weight vector is taken as $\mathbf{a}(\theta_0)$. Thus, the normalized array response at θ_i can be arbitrarily set (from 0 to $+\infty$) by tuning β. However, as shall be discussed later, given a weight vector other than $\mathbf{a}(\theta_0)$, the normalized array response at an angle cannot be set to an arbitrarily large value. More precisely, the normalized array response should be no larger than one. As a matter of fact, it is a common sense that, given the reference direction θ_0, it is implicitly required that the normalized array responses at any other directions are no larger than one, i.e.,

$$0\le L(\theta,\theta_0)\le 1. \tag{19}$$

In the sequel, the normalized array response will be restricted to this condition.

C. The Proposed WORD Scheme

In this subsection, the WORD algorithm for a given weight vector is presented based on the previous discussions. According to (9), it is known that $\mathbf{w}_\star$ is updated from $\mathbf{w}_0$ with an appropriate value of β, and hence, we treat $\mathbf{w}_\star$ and $\mathbf{w}_0$ as the current weight vector and the previous weight vector, respectively. This motivates us to generalize such a procedure to adjust the response of a given weight vector. More exactly, suppose that the previous weight vector is $\mathbf{w}_{(k-1)}$, the desired weight vector $\mathbf{w}_{(k)}$ which adjusts the response at a direction θ_k to a certain level, say ρ_k, can be updated as

$$\mathbf{w}_{(k)} = \begin{bmatrix} \mathbf{w}_{(k-1)\perp} & \mathbf{w}_{(k-1)\parallel} \end{bmatrix} \begin{bmatrix} 1 & \beta \end{bmatrix}^{\mathrm{T}} \tag{20}$$

where $\mathbf{w}_{(k-1)\perp}$ and $\mathbf{w}_{(k-1)\parallel}$ are defined according to (10) as

$$\mathbf{w}_{(k-1)\perp} = \mathbf{P}^{\perp}_{[\mathbf{a}(\theta_k)]}\mathbf{w}_{(k-1)} \tag{21a}$$

$$\mathbf{w}_{(k-1)\parallel} = \mathbf{P}_{[\mathbf{a}(\theta_k)]}\mathbf{w}_{(k-1)} \tag{21b}$$

and β can be determined from the following identity

$$L_{(k)}(\theta_k, \theta_0) = \frac{|\mathbf{w}^{\mathrm{H}}_{(k)}\mathbf{a}(\theta_k)|^2}{|\mathbf{w}^{\mathrm{H}}_{(k)}\mathbf{a}(\theta_0)|^2} = \rho_k. \tag{22}$$

In order to solve the above problem with respect to β, we first substitute (20) into the numerator and denominator of (22), and get

$$|\mathbf{w}^{\mathrm{H}}_{(k)}\mathbf{a}(\theta_\tau)|^2 = \begin{bmatrix} 1 & \beta \end{bmatrix} \begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_\tau) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_\tau) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_\tau) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \tag{23}$$

where $\tau \in \{0, k\}$, $\mathbf{w}_{\perp} \triangleq \mathbf{w}_{(k-1)\perp}$ and $\mathbf{w}_{\parallel} \triangleq \mathbf{w}_{(k-1)\parallel}$ are defined for notational simplicity.

As a consequence, after some manipulations, the problem in (22) can be rewritten as

$$\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{z} = 0 \tag{24}$$

where $\mathbf{z} \triangleq [1 \ \ \beta]^{\mathrm{T}}$, $\mathbf{B}$ is a 2×2 Hermitian matrix given by

$$\begin{aligned} \mathbf{B} = &\begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_k) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_k) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_k) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_k) \end{bmatrix}^{\mathrm{H}} - \rho_k \begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_0) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_0) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_0) \\ \mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_0) \end{bmatrix}^{\mathrm{H}} \\ = &\begin{bmatrix} -\rho_k|\mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_0)|^2 & -\rho_k\mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_0)\mathbf{a}^{\mathrm{H}}(\theta_0)\mathbf{w}_{\parallel} \\ -\rho_k\mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_0)\mathbf{a}^{\mathrm{H}}(\theta_0)\mathbf{w}_{\perp} & |\mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_k)|^2 - \rho_k|\mathbf{w}^{\mathrm{H}}_{\parallel}\mathbf{a}(\theta_0)|^2 \end{bmatrix}. \end{aligned} \tag{25}$$

where the identity $\mathbf{w}^{\mathrm{H}}_{\perp}\mathbf{a}(\theta_k) = 0$ has been used.

As shown in Appendix B, if $0 \leq \rho_k \leq 1$ and $\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k) > |\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|$, then (24) can be analytically solved with two solutions given by

$$\beta_a = \frac{-\Re(\mathbf{B}(1,2)) + d}{\mathbf{B}(2,2)} \tag{26a}$$

$$\beta_b = \frac{-\Re(\mathbf{B}(1,2)) - d}{\mathbf{B}(2,2)} \tag{26b}$$

where $d = \sqrt{\Re^2(\mathbf{B}(1,2)) - \mathbf{B}(1,1)\mathbf{B}(2,2)}$ and $\Re(\cdot)$ represents the real part of a complex number.

Note that in general we have $\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k) \geq |\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|$ and the normalized response should be no larger than 1, i.e., $0 \leq \rho_k \leq 1$, as mentioned earlier. Up to now, the new WORD scheme has been established.

IV. Pattern Synthesis via WORD

It is noticed that the WORD scheme can only deal with the control of the response at a single direction. In this section, we shall show how this scheme can be applied to synthesize a pattern for an arbitrary array, which involves the control of the sidelobe and/or mainlobe regions rather than a single direction.

Briefly speaking, the desired pattern is obtained by successively adjusting the response levels at the directions where the specifications do not meet. More precisely, let $L_d(\theta, \theta_0)$ be the desired pattern. An initial pattern is firstly obtained by setting the weight vector as $\mathbf{w}_{(0)}$ (not necessarily equals to $\mathbf{a}(\theta_0)$), and an angle θ_1, at which the response level requires adjustment, is selected by comparing the initial pattern with the desired one. Next, the WORD scheme is applied to modify the weight vector $\mathbf{w}_{(0)}$ to $\mathbf{w}_{(1)}$, by setting the desired response level at θ_1 as $L_d(\theta_1, \theta_0)$. Similarly, by comparing $L_{(1)}(\theta, \theta_0)$ with $L_d(\theta, \theta_0)$, a second angle θ_2, at which the response is needed to be adjusted, is selected. An updated weight vector $\mathbf{w}_{(2)}$ can thus be achieved via WORD. The above procedure is carried out successively once a satisfactory array pattern has been obtained.

In summary, the proposed WORD-based pattern synthesis technique includes the following main steps:

1) Give the initial angle θ_0, weight vector $\mathbf{w}_{(0)}$ and desired pattern $L_d(\theta, \theta_0)$, set $k = 1$.
2) Determine the angle θ_k where the response level needs adjustment.
3) Obtain the weight vector $\mathbf{w}_{(k)}$ by selecting an appropriate value of β to achieve $L_{(k)}(\theta_k, \theta_0) = L_d(\theta_k, \theta_0)$.
4) Check whether the pattern $L_{(k)}(\theta, \theta_0)$ is satisfactorily synthesized. If not, set $k = k + 1$ and repeat steps 2 and 3, otherwise, return the outputs $\mathbf{w}_{(k)}$ and $L_{(k)}(\theta, \theta_0)$.

It is noted that two key problems should be solved in the WORD-based pattern synthesis approach. The first one is how to select the angle θ_k where the response needs adjustment in the k-th step. The other one is how to select one of the two β's (i.e., β_a and β_b) derived above to finally obtain a unique weight vector with better performance. These two problems will be discussed in the following two subsections.

A. Selection of θ_k

In our proposed approach, θ_k is determined by comparing the temporarily obtained response with the desired one. In particular, for sidelobe synthesis, θ_k is selected as the one where $L_{(k-1)}(\theta, \theta_0)$ exceeds the most from the desired $L_d(\theta, \theta_0)$, i.e.,

$$\theta_k = \arg\max_{\theta \in \tilde{\Omega}^{(k)}_s} \left(L_{(k-1)}(\theta, \theta_0) - L_d(\theta, \theta_0) \right) \tag{27}$$

where $\tilde{\Omega}^{(k)}_s = \{\theta | L_{(k-1)}(\theta, \theta_0) > L_d(\theta, \theta_0), \theta \in \Omega_s\}$ and Ω_s denotes the sidelobe region. On the other hand, for mainlobe synthesis, θ_k is selected as

$$\theta_k = \arg\max_{\theta \in \Omega_m} |L_{(k-1)}(\theta, \theta_0) - L_d(\theta, \theta_0)| \tag{28}$$

where Ω_m denotes the mainlobe region.

It should be mentioned that when both the sidelobe and mainlobe regions need to be synthesized, we first control the response

TABLE II
SUMMARY OF WORD-BASED PATTERN SYNTHESIS

Input	$k=1$, θ_0, $\mathbf{w}_{(0)}$, Ω_m, Ω_s, $L_d(\theta,\theta_0)$, $L_{(0)}(\theta,\theta_0)$
Step 1.	**Sidelobe control**: Find out θ_k in sidelobe region using (27). **Mainlobe control**: Determine θ_k using (28).
Step 2.	Determine $\mathbf{B}$ in (25) and calculate β_a and β_b using (26).
Step 3.	Select $\beta_\star$ which minimizes $F(\beta)$, obtain $\mathbf{w}_{(k)}$ as (30) and calculate $L_{(k)}(\theta,\theta_0)=\lvert\mathbf{w}_{(k)}^{\mathrm{H}}\mathbf{a}(\theta)\rvert^2/\lvert\mathbf{w}_{(k)}^{\mathrm{H}}\mathbf{a}(\theta_0)\rvert^2$.
Step 4.	Check whether the pattern is satisfactorily synthesized. If not, set $k=k+1$ and go to **Step 1** to repeat the above procedures, otherwise, return the outputs.
Output	$\mathbf{w}_{(k)}$ and $L_{(k)}(\theta,\theta_0)$.

in the mainlobe region until the resultant pattern meets requirement. Then, we proceed to the sidelobe synthesis. This procedure may be performed repeatedly until satisfactory patterns have been synthesized in both mainlobe and sidelobe regions.

B. Selection of β

From the last section, it is known that both β_a and β_b in (26) lead to the desired response level ρ_k at θ_k in the k-th step. However, a more appropriate β should be picked from them such that the pattern can be properly synthesized. With such consideration, it is required that the variation of the synthesized patterns between the k-th and $(k-1)$-th steps should be minimized. To this end, the variation of the pattern between two iterations is measured as

$$\begin{aligned} F(\beta) &= \left\| \mathbf{P}^{\perp}_{[\mathbf{w}_{(k-1)}]} \frac{\mathbf{w}_{(k)}}{\|\mathbf{w}_{(k)}\|_2} \right\|_2^2 \\ &= \left\| \mathbf{P}^{\perp}_{[\mathbf{w}_{(k-1)}]} \frac{\mathbf{w}_{(k-1)\perp} + \beta \mathbf{w}_{(k-1)\|}}{\sqrt{\|\mathbf{w}_{(k-1)\perp}\|_2^2 + \beta^2 \|\mathbf{w}_{(k-1)\|}\|_2^2}} \right\|_2^2. \end{aligned} \tag{29}$$

Hence, either β_a or β_b is chosen such that $F(\beta)$ is minimized. Let $\beta_\star$ be the solution, then the weight vector $\mathbf{w}_{(k)}$ can be computed as

$$\mathbf{w}_{(k)} = \left[\, \mathbf{w}_{(k-1)\perp}\ \mathbf{w}_{(k-1)\|} \,\right] \left[\, 1\ \beta_\star \,\right]^{\mathrm{T}}. \tag{30}$$

It is seen that, different from the parameter selection means adopted in [42], the parameter here is real-valued and can be simply determined. Finally, the WORD-based approach for pattern synthesis is summarized in Table II.

V. COMPARISONS OF A^2RC AND WORD

In the A^2RC method [42], given the previous weight $\mathbf{w}_{(k-1)}$ and an angle θ_k where the array response needs adjustment, the weight vector is updated as

$$\mathbf{w}_{(k)} = \mathbf{w}_{(k-1)} + \mu \mathbf{a}(\theta_k) \tag{31}$$

where μ is the parameter to be determined. More specially, for a certain pattern requirement (e.g., (22)), it has been shown in [42] that μ locates in a specific set $\mathbb{D}$, i.e.,

$$\mu \in \mathbb{D} \triangleq \{\mu \big| [\Re(\mu) \quad \Im(\mu)]^{\mathrm{T}} \in \mathbb{C}_\mu\} \tag{32}$$

where $\mathbb{C}_\mu$ is the trajectory set of $[\Re(\mu)\ \ \Im(\mu)]^{\mathrm{T}}$ to realize the design requirement (22), $\Im\{\cdot\}$ denotes the imaginary part of a complex-valued number. The set $\mathbb{C}_\mu$, as discussed in [42], is a circle with specific center point and radius.

In order to determine μ from $\mathbb{D}$, the variation of the synthesized patterns between the k-th and $(k-1)$-th steps should be minimized and the variation is defined in [42] as $\int_{\theta\neq\theta_k} \big| L_{(k)}(\theta,\theta_0) - L_{(k-1)}(\theta,\theta_0) \big| d\theta$ whose discretized version is given by

$$J = \frac{1}{I}\sum_{i=1}^{I} \big| L_{(k)}(\theta_i,\theta_0) - L_{(k-1)}(\theta_i,\theta_0) \big| \tag{33}$$

where I denoting the number of sampling points in the angle sector. Theoretically, a global research has to be performed to find out the optimal value of μ to minimize the variation. Since this is computationally intensive, an alternative strategy to select μ was proposed in [42] with the aid of its geometrical distribution as

$$\mu_\star = \arg\min_{\mu\in\mathbb{D}} \ |\mu| \triangleq \mu_a. \tag{34}$$

However, we have to emphasize that in this strategy it cannot be rigorously derived that the ultimate selection of μ in (34) could produce an optimal pattern with minimum variation. Furthermore, as shall be demonstrated later, the determination of μ as (34) may even maximize the pattern variation and eventually lead a distorted pattern in certain cases, which was not well investigated in [42].

Unlike the A^2RC algorithm, in the WORD algorithm, there are only two candidates, i.e., β_a and β_b in (26), to achieve the requirement (22). Moreover, a new criterion is developed in (29) to measure the pattern variation, and a simple testing problem is devised to determine the final $\beta_\star$, which can avoid the pattern distortion problem encountered in A^2RC.

In brief, the main differences between the proposed WORD method and A^2RC method include:

- The variable β to be optimized in WORD is real-valued, whereas the variable μ in A^2RC is complex-valued.
- There are only two candidates for β to be picked up in WORD, whereas in A^2RC, the parameter μ is determined from a set with infinite candidates.
- A simple testing problem can be used to determine β in WORD, whereas in A^2RC a global search is theoretically required to obtained the optimal μ, although a geometrical approach is empirically applied.
- A more simple cost function (29) is introduced in WORD as compared to the one (33) in A^2RC to choose the desirable values of the parameters.
- The determination of β in WORD is theoretically guaranteed, whereas the determination of μ via (34) is only an empirical approach.

VI. NUMERICAL RESULTS

In this section, representative numerical examples are carried out. Without specification, the quiescent weight vector $\mathbf{a}(\theta_0)$ will be employed as the initial weight $\mathbf{w}_{(0)}$ in the following

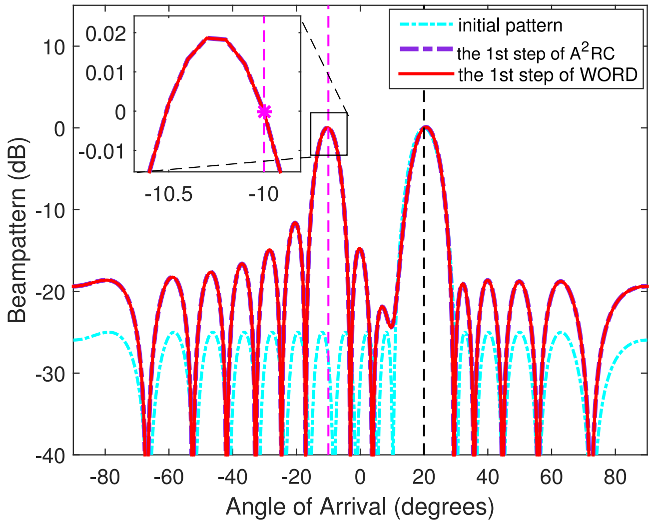

Fig. 3. Achieve to $L_{(1)}(\theta_1, \theta_0) = 0$ dB from the Chebyshev weight.

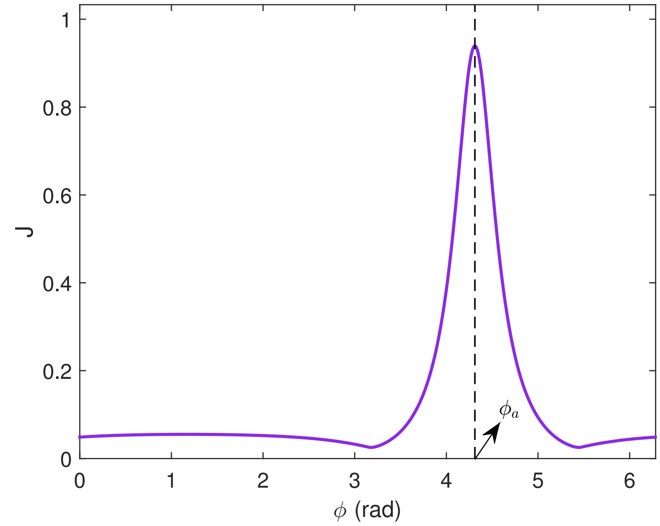

Fig. 4. J versus ϕ in the second step of response control.

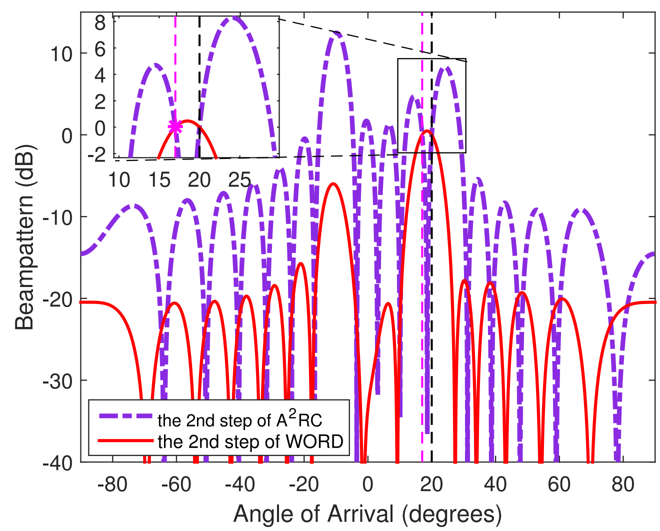

Fig. 5. Achieve to $L_{(2)}(\theta_2, \theta_0) = 0$ dB on the basis of the first step.

simulations. For comparison purpose, the synthesized patterns of the A^{2}RC method in [42] and optimization based method in [29] or [31] will also be demonstrated. More specially, if there is no lower bound constraint on the desired pattern, the convex programming (CP) method in [29] will be carried out. Otherwise, the SDR method in [31] will be executed due to the non-convexity of the problem.

A. *Response Control of a Uniformly Spaced Linear Array*

In this example, a ULA with 16 isotropic elements is considered. The beam center is fixed at $\theta_0 = 20°$. To show the flexibility of the proposed WORD approach, we set $\mathbf{w}_{(0)}$ as the Chebyshev weight vector with a -25 dB uniform sidelobe level. On this basis, we successively adjust the normalized response levels at $\theta_1 = -10°$ and $\theta_2 = 17$ to be 0 dB and compare the performance of the proposed WORD approach with that of A^{2}RC. Additionally, to measure the function J in (33), we uniformly sample the region $[-90°, 90°]$ every $0.1°$ and hence obtain 1801 discrete points, i.e., $I = 1801$.

In the first step, we can figure out that $\beta_a = 27.1619$, $\beta_b = -25.4210$ and further $F(\beta_a) = 0.4590$, $F(\beta_b) = 0.4988$. Since $F(\beta_a) < F(\beta_b)$, it is implied that β_a should be selected. When considering the variation of J in (33), we have $J(\beta_a) = 0.0479 < J(\beta_b) = 0.0488$, which is consistent with the relationship between $F(\beta_a)$ and $F(\beta_b)$. For A^{2}RC, it can be calculated that $\mu_a = 0.6515 - j0.2880$. Interestingly, we can verify that the WORD method and A^{2}RC method actually result the same weight vector $\mathbf{w}_{(1)}$. Fig. 3 shows the resultant patterns of these two approaches. It can be seen that the obtained patterns of WORD and A^{2}RC are coincident. More importantly, either of them achieves $L_{(1)}(\theta_1, \theta_0) = 0$ dB with no pattern distortion occurred.

On the basis of the first step, the second step is carried out to realize $L_{(2)}(\theta_2, \theta_0) = 0$ dB. In this step, we have $\beta_a = 2.5907$, $\beta_b = -0.3520$, $F(\beta_a) = 0.1959$ and $F(\beta_b) = 0.4553$. Obviously, the β_a should be selected since $F(\beta_a) < F(\beta_b)$. When considering J in (33), we can calculate that $J(\beta_a) = 0.0551 < J(\beta_b) = 0.9392$, which is still consistent with $F(\beta_a) < F(\beta_b)$. On the other hand, for the A^{2}RC method it can be figured out that $\mu_a = -0.2827 - j0.6668$ and $\phi_a = 4.3114$ (defined in Eq. (51) of [42]). To show the irrationality of selecting μ_a, the curve of J versus ϕ (see Eq. (40) of [42]) is depicted in Fig. 4. It is seen that J achieves the maximum at $\phi = \phi_a$. This observation was unfortunately ignored in [42] and implies that the selection of μ_a actually results a pattern with the worst performance. Fig. 5 shows the resulting patterns in the above scenario. It is clearly seen that the response levels can be precisely adjusted to be the desired values for both approaches. However, it is noticed that the obtained pattern of A^{2}RC is severely distorted, and a good pattern is difficult to re-obtain by continuing the synthesis procedure. While for the proposed WORD approach, a preferable parameter is selected to guarantee the outstanding performance of the resulting pattern.

B. *Uniform Sidelobe Synthesis for ULA*

Consider a 11-element ULA with isotropic elements. The desired pattern steers at $\theta_0 = 20°$ with uniform sidelobe level less

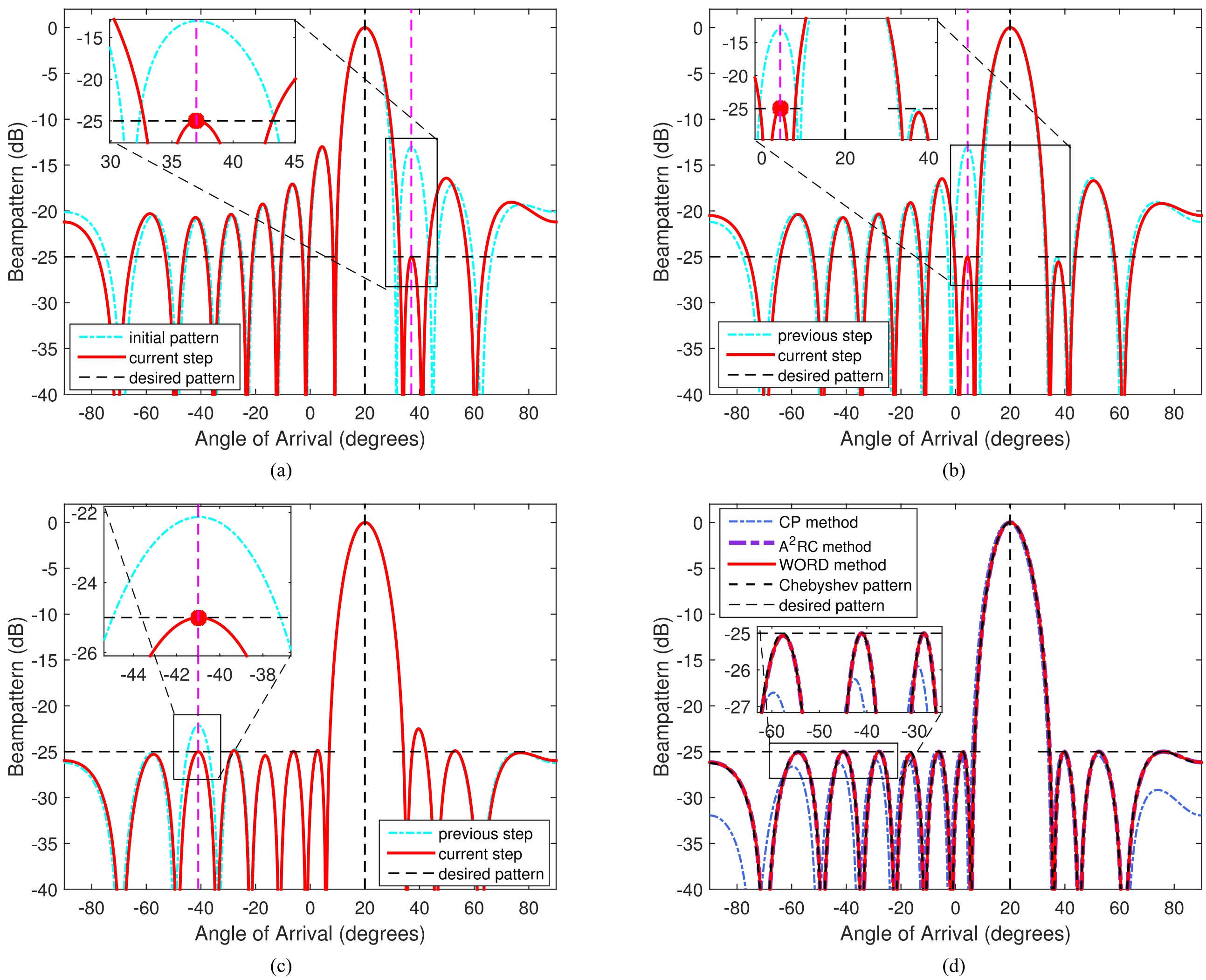

Fig. 6. Synthesis procedure of uniform-sidelobe pattern using a ULA. (a) Synthesized pattern at the first step. (b) Synthesized pattern at the second step. (c) Synthesized pattern at the 10-th step. (d) Comparison of the synthesized pattern after 15 steps with other methods.

than -25 dB. From the initial pattern, a single angle $\theta_1 \approx 37^\circ$, which locates in the sidelobe region and at which the response level exceeds most than its corresponding desired level, is selected. On this basis, a weight vector $\mathbf{w}_{(1)}$ can be obtained via WORD to adjust the normalized level at θ_1 to -25 dB. Fig. 6(a) shows the initial pattern and the synthesized pattern $L_{(1)}(\theta, \theta_0)$ after the first step.

In the second step, the angle $\theta_2 \approx 5^\circ$ is selected since $L_{(1)}(\theta_2, \theta_0)$ exceeds most above -25 dB in sidelobe region. Fig. 6(b) depicts the two patterns $L_{(1)}(\theta, \theta_0)$ and $L_{(2)}(\theta, \theta_0)$. It is seen that the response at θ_2 has been accurately adjusted to -25 dB, and the response level at θ_1 still meets the specification.

Fig. 6(c) plots the synthesized pattern at the 10-th step. It is observed that all sidelobe levels of the resultant pattern are close to the desired values.

Fig. 6(d) depicts the obtained pattern at the 15-th step. For comparison purpose, the resulting patterns of Chebyshev method, A^{2}RC method in [42] and CP method in [29] are also displayed. Note that the runtimes of WORD, A^{2}RC and CP methods are respectively 0.04 s, 0.04 s and 4.02 s. Hence, the WORD is more computationally efficient than the CP method. Interestingly, the obtained patterns of WORD and A^{2}RC are same in this specific scenario, and either of them has nearly no difference between the Chebyshev pattern. Additionally, it is seen that the resulting pattern of the CP method is somewhat different from that of the WORD algorithm. This is because the response level in the CP method is only required to be no larger than (but not exactly equal to) the prescribed value.

To explore the convergence of the proposed approach, let us define D_k as the maximum response deviation within the sector $\tilde{\Omega}_s^{(k)}$ at the kth step, i.e.,

$$D_k \triangleq \max_{\theta \in \tilde{\Omega}_s^{(k)}} \left(L_{(k)}(\theta, \theta_0) - L_d(\theta, \theta_0) \right). \tag{35}$$

Fig. 7 plots the curve of D_k against the iteration number k. Clearly, it is seen that D_k decreases along with the increase of iterations. Moreover, the change of D_k is ignorable after 11 steps. Therefore, we terminate the synthesis process after 15 iterations in this example. As a matter of fact, in all examples

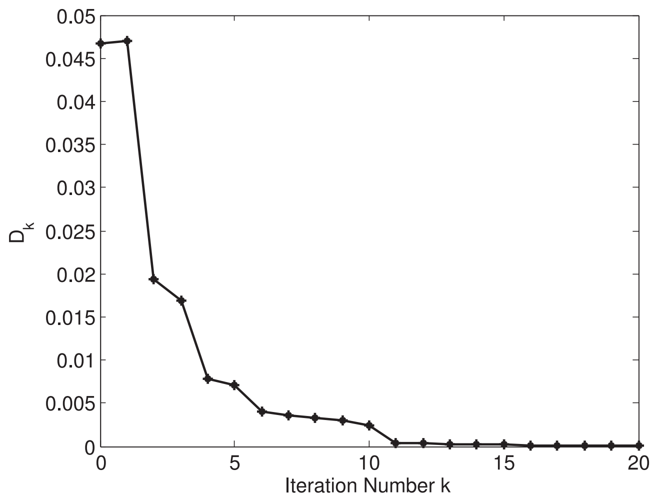

Fig. 7. Maximum response deviation D_k versus the iteration number.

TABLE III
ELEMENT LOCATIONS OF NONUNIFORMLY SPACED LINEAR ARRAY AND WEIGHTS OBTAINED BY THE PROPOSED METHOD

n	$x_n(\lambda)$	w_n	n	$x_n(\lambda)$	w_n
1	0.00	$0.3729e^{+j0.0629}$	9	3.99	$0.5389e^{+j1.0138}$
2	0.47	$0.2710e^{+j0.4647}$	10	4.48	$0.9496e^{+j1.4446}$
3	1.01	$0.1708e^{-j2.1372}$	11	4.96	$0.4097e^{+j2.0121}$
4	1.47	$0.6945e^{-j1.6171}$	12	5.43	$0.4452e^{-j0.6784}$
5	1.97	$0.5003e^{-j1.0681}$	13	5.94	$0.7115e^{-j0.1635}$
6	2.54	$0.5006e^{+j2.6699}$	14	6.49	$0.1974e^{+j0.2931}$
7	3.06	$1.0000e^{+j3.1288}$	15	6.98	$0.3151e^{-j2.3056}$
8	3.53	$0.3728e^{-j2.5715}$	16	7.46	$0.3057e^{-j1.7021}$

we tested in this work, it is found that the proposed method converges to satisfactory solutions.

C. Multibeam Pattern Synthesis for Nonuniformly Spaced Linear Array

In this example, multibeam pattern synthesis for the 16-element nonuniformly spaced linear array is considered. The element locations are given in Table III. We assume that the two beams steering at 30° and -10°, respectively. Here, we take $\mathbf{a}(30^\circ)$ as the initial weight, and then apply the proposed WORD algorithm to control the response at -10° to 0 dB by setting $L_{(1)}(-10^\circ, 30^\circ) = 0$ dB. On this basis, the proposed WORD based pattern synthesis method is utilized to adjust sidelobe response to be lower than -25 dB. After implementing 50 steps, a satisfactory pattern is obtained and the corresponding weightings are listed in Table III.

The synthesized patterns of the WORD, $\mathrm{A^2RC}$ (with a same iteration number as WORD) and CP methods are shown in Fig. 8. It is clearly seen that the resultant pattern of WORD is well synthesized in both sidelobe region and mainlobe region. However, the obtained pattern of $\mathrm{A^2RC}$ is somewhat irregular, due to its irrational determination of parameter μ. The resulting pattern of CP method meets the design requirement. However, since it does not attempt to exactly adjust the response level to a specific value, the synthesized pattern is more irregular.

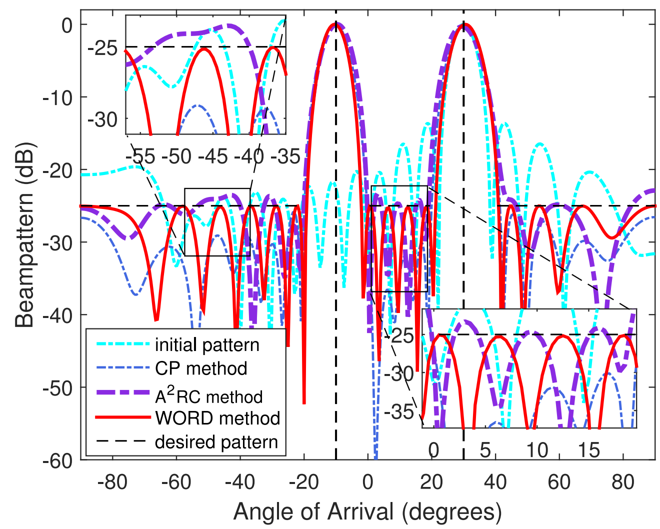

Fig. 8. Synthesized multibeam patterns for a nonuniformly spaced linear array.

TABLE IV
PARAMETERS OF THE NONISOTROPIC RANDOM ARRAY AND THE OBTAINED WEIGHTINGS BY WORD APPROACH

n	$x_n(\lambda)$	$l_n(\lambda)$	ζ_n(deg)	w_n
1	0.00	0.30	0.0	$0.0927e^{-j0.1413}$
2	0.45	0.25	–4.0	$0.0723e^{+j0.2287}$
3	0.95	0.24	5.0	$0.1668e^{-j1.3178}$
4	1.50	0.20	–32	$0.1923e^{-j2.3039}$
5	2.04	0.26	–3.2	$0.0999e^{+j1.1799}$
6	2.64	0.27	10	$0.4253e^{+j0.7218}$
7	3.09	0.23	1.0	$0.8710e^{+j1.0902}$
8	3.55	0.24	–10	$1.0000e^{+j0.8067}$
9	4.05	0.25	0.0	$0.7939e^{+j0.8051}$
10	4.55	0.21	7.0	$0.3750e^{+j1.0942}$
11	5.06	0.20	5.0	$0.1188e^{-j2.1384}$
12	5.50	0.20	5.0	$0.3864e^{-j2.1251}$
13	6.01	0.29	4.0	$0.1890e^{-j1.9473}$
14	6.53	0.20	5.0	$0.1398e^{-j0.4987}$
15	7.07	0.26	–9.0	$0.1425e^{+j0.5424}$
16	7.52	0.21	7.0	$0.0928e^{+j0.3122}$
17	8.00	0.25	10	$0.0585e^{+j1.0631}$
18	8.47	0.21	6.0	$0.1749e^{-j2.3102}$
19	8.98	0.20	–8.0	$0.0895e^{-j2.7192}$
20	9.53	0.26	0.0	$0.0313e^{-j1.7434}$
21	10.01	0.25	5.0	$0.0663e^{+j1.0571}$

D. Pattern Synthesis for Nonisotropic Linear Random Array

To further examine the performance of the proposed method for pattern synthesis, a 21-element nonisotropic linear random array (see e.g., [42]–[44]) is considered. The pattern of the nth element is given by

$$g_n(\theta) = \frac{\cos(\pi l_n \sin(\theta + \zeta_n)) - \cos(\pi l_n)}{\cos(\theta + \zeta_n)} \qquad (36)$$

where ζ_n and l_n represent the orientation and length of the element, respectively. More details of the array can be found in Table IV, where the element positions (in wavelength) also specialized. The desired pattern has flat-top mainlobe and broad-

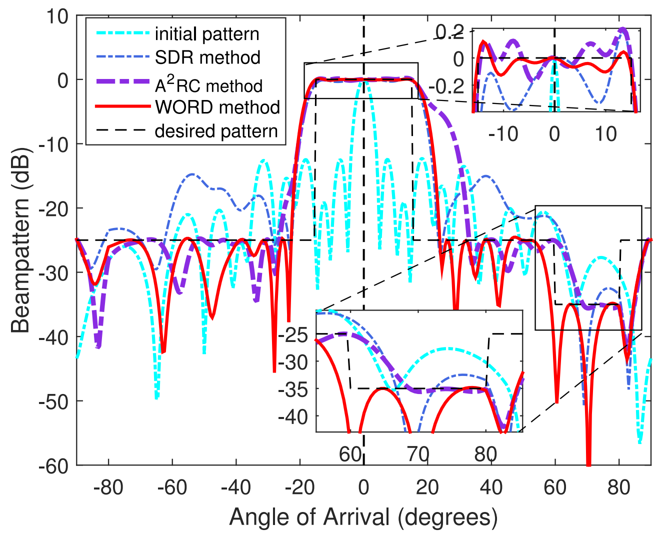

Fig. 9. Synthesized pattern with flat-top mainlobe and broad-notch sidelobe for a nonisotropic random array.

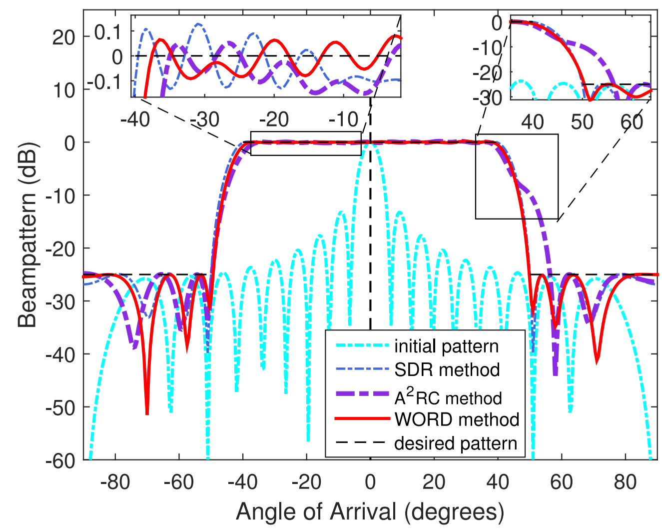

Fig. 10. Synthesized pattern with flat-top mainlobe.

notch sidelobe. Specifically, all response levels in the mainlobe $[-15^{\circ}, 15^{\circ}]$ are expected to be 0 dB. The upper level is -35 dB in the sidelobe region $[60^{\circ}, 80^{\circ}]$ and -25 dB in the rest of the sidelobe region.

In this scenario, constraints are set to simultaneously control the upper and lower bounds of the mainlobe. Therefore, the CP method [29] is not applicable owing to the fact that the lower bound constraint is non-convex. For this reason, the SDR method [31], which is also based on convex optimization but overcoming the drawback of [29], is conducted and tested. Fig. 9 plots the initial pattern and the synthesized patterns of different approaches. The resulting weightings of the proposed WORD approach (after carrying out 450 steps of iteration) are shown in Table IV. Obviously, it can be noticed from Fig. 9 that although the initial pattern is considerably different from the desired one, the proposed WORD-based approach successfully synthesizes a satisfactory pattern which has a flat-top mainlobe with ripple being less than 0.2 dB and sidelobes as desired. On the other hand, from Fig. 9 we know that the obtained patterns of the A^2RC method and SDR method are distorted. Hence, either of these two approaches fails to synthesize a desirable pattern in this case. It should be pointed out that the performance of the SDR method depends on the specific problem. It can perform well in certain cases as shown in the next example.

E. Pattern Synthesis for Isotropic Uniform Linear Array

Following [31], we consider a 20-element ULA (with inter-elment spacing being 0.45 wavelengths) to design a pattern having a flat-top mainlobe and a uniform sidelobe. Specifically, all response levels in the mainlobe $[-40^{\circ}, 40^{\circ}]$ are expected to be 0 dB and the upper level is -25 dB in the sidelobe region. The initial pattern and the synthesized patterns of different approaches are depicted in Fig. 10. It is seen that both the proposed WORD-based approach and SDR method successfully synthesizes a satisfactory pattern, which has a flat-top mainlobe with ripple being less than 0.1 dB and sidelobes as desired. However, it is experimentally found that, to complete the synthesis procedure, the SDR method requires about one minute, which is much longer than that of the proposed WORD method.

F. Pattern Synthesis for Two-Dimensional Array

In order to demonstrate the wide applicability of the proposed WORD-based approach. An example of pattern synthesis for a two-dimensional array is carried out. Without loss of generality, we consider a rectangular array composed of 16×16 isotropic elements which are spaced by half a wavelength. Fig. 11(a) shows the desired pattern, where $u = \sin(\theta_e)\cos(\theta_a)$, $v = \sin(\theta_e)\sin(\theta_a)$, and θ_e and θ_a stand for elevation and azimuth angles, respectively. The desired pattern steers at $(u_0, v_0) = (0.3, 0.3)$ with a mainlobe region $\Theta_M = \{(u,v)\big||u-u_0| + |v-v_0| \leq 0.3\}$. The upper level of the desired pattern is -35 dB in the sidelobe region $\Theta_S = \{(u,v)\big| -0.8 \leq u \leq -0.5\}$ and -15 dB in the rest of the sidelobe region.

Fig. 11(b) displays the resulting synthesized pattern of A^2RC approach after carrying out 3000 steps. Notice that the mainlobe region of the obtained pattern is fluctuant to some extent. Fig. 11(c) gives the synthesized pattern of the proposed WORD approach after executing the same number of iteration steps, and Fig. 11(d) plots its top view. It can be clearly seen that the resultant mainlobe beampattern is quite flat, meanwhile, the sidelobe region of synthesized pattern meets the qualification.

To illustrate the superiority of WORD more clearly, Fig. 12 shows the beampattern in u direction for a fixed $v = v_0$ and Fig. 13 plots the beampattern in v direction for a fixed $u = u_0$. From these two figures, it can be observed that the mainlobe pattern of A^2RC method is far from satisfaction. For the devised WORD approach, the obtained ripple in mainlobe region of the resultant pattern is less than 0.2 dB and the sidelobe level meets the requirement well. Therefore, the preference of the proposed WORD method, especially when comparing with A^2RC,

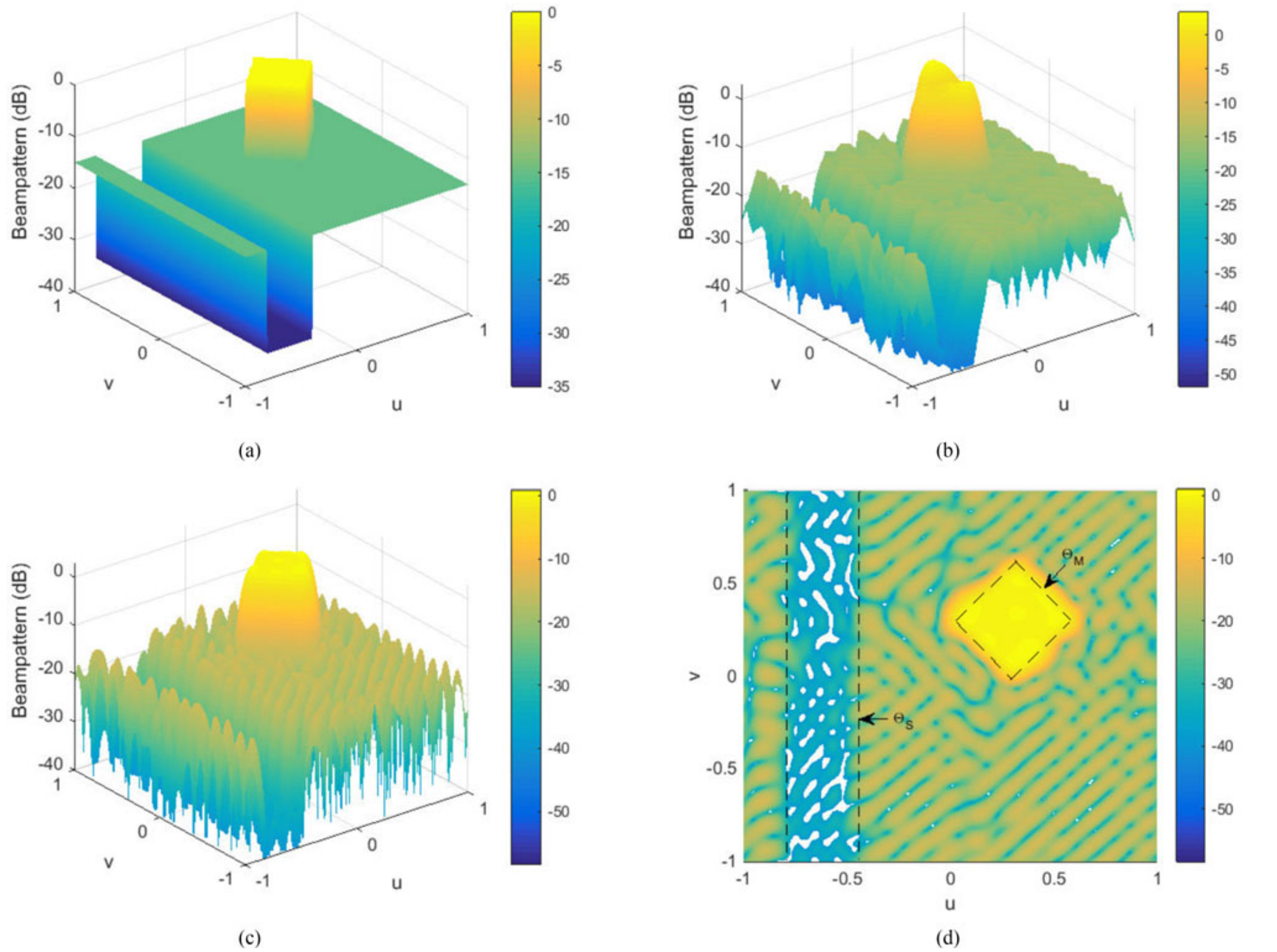

Fig. 11. 2-D pattern synthesis with a 16×16 isotropic elements planar array. (a) The desired 2-D shape. (b) The synthesized beampattern via A^{2}RC approach. (c) The synthesized beampattern via WORD approach. (d) Top view of the 2-D synthesized pattern of WORD method.

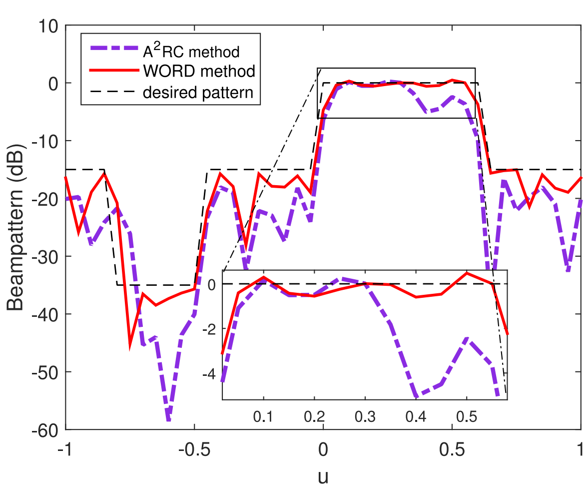

Fig. 12. Beampattern in u direction for a fixed $v = v_0$.

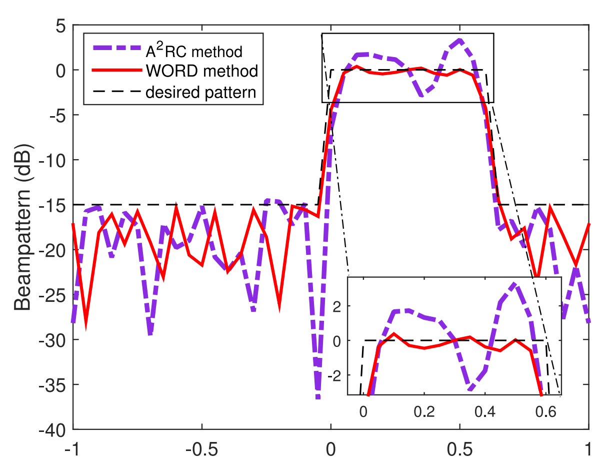

Fig. 13. Beampattern in v direction for a fixed $u = u_0$.

is obvious. Note that the SDR method in [31] is not presented in this case, since it turns out to be time-consuming and even unsolvable due to the large number of constraints in the formulated optimization problem.

VII. Conclusions

In this paper, a novel scheme called weight vector orthogonal decomposition (WORD) has been developed to flexibly and accurately control the array response level. In this algorithm, an analytical expression of the weight vector has been derived to adjust the beampattern response at a given angle to its preassigned level. Moreover, it has been successfully employed to synthesize desired patterns for arbitrary arrays. The proposed WORD-based pattern synthesis approach successively adjusts the response levels to their desired values until the whole pattern is successfully synthesized. A more feasible criterion has been introduced to guarantee the optimality of synthesized pattern in each step. The difference between the proposed method and previous A^{2}RC method has been summarized. The effectiveness and superiority of the WORD scheme for array response control at a single direction and pattern synthesis have been validated by various examples including linear and rectangular arrays. It is worth mentioning that the WORD algorithm is vulnerable to array imperfections, which have not been considered in this work, and we shall consider its extension for robust pattern synthesis under uncertainties in our future work.

Appendix A
Detailed Derivation of (18)

Firstly, substituting $\mathbf{w}_\star = \mathbf{w}_{(0)\perp} + \beta\mathbf{w}_{(0)\parallel}$ into $L_\star(\theta_i,\theta_0)$ and recalling the identity $\mathbf{w}_{(0)\perp}^{\mathrm{H}}\mathbf{a}(\theta_i) = 0$, one gets

$$L_\star(\theta_i,\theta_0) = \frac{|\mathbf{w}_\star^{\mathrm{H}}\mathbf{a}(\theta_i)|^2}{|\mathbf{w}_\star^{\mathrm{H}}\mathbf{a}(\theta_0)|^2} = \frac{\beta^2\left|\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_i)\right|^2}{\left|\mathbf{w}_{(0)\perp}^{\mathrm{H}}\mathbf{a}(\theta_0) + \beta\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0)\right|^2}. \tag{37}$$

Note that according to (10), we have

$$\mathbf{w}_{(0)\parallel} = \frac{\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i)\mathbf{a}(\theta_0)}{\mathbf{a}^{\mathrm{H}}(\theta_i)\mathbf{a}(\theta_i)}. \tag{38}$$

It can be simply verified that $\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0)$ is real and positive. Thus, dividing $(\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0))^2$ on both numerator and denominator of (37) yields

$$L_\star(\theta_i,\theta_0) = \frac{|\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_i)|^2}{(\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0))^2} \cdot \frac{\beta^2}{\left|\frac{\mathbf{w}_{(0)\perp}^{\mathrm{H}}\mathbf{a}(\theta_0)}{\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0)} + \beta\right|^2}. \tag{39}$$

Owing to the fact that $\mathbf{a}(\theta_0) = \mathbf{w}_{(0)} = \mathbf{w}_{(0)\perp} + \mathbf{w}_{(0)\parallel}$ and $\mathbf{w}_{(0)\perp}^{\mathrm{H}}\mathbf{w}_{(0)\parallel} = 0$, we have

$$\frac{\mathbf{w}_{(0)\perp}^{\mathrm{H}}\mathbf{a}(\theta_0)}{\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0)} = \frac{\|\mathbf{w}_{(0)\perp}\|_2^2}{\|\mathbf{w}_{(0)\parallel}\|_2^2} = -\beta_p \tag{40}$$

where β_p is defined above (16). On this basis, $L_\star(\theta_i,\theta_0)$ can be further simplified as

$$L_\star(\theta_i,\theta_0) = \mu(\theta_i,\theta_0)\cdot\frac{\beta^2}{(\beta-\beta_p)^2} \tag{41}$$

where $\mu(\theta_i,\theta_0) \triangleq |\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_i)|^2/(\mathbf{w}_{(0)\parallel}^{\mathrm{H}}\mathbf{a}(\theta_0))^2 > 0$. Thus, the partial derivative of $L_\star(\theta_i,\theta_0)$ with respect to β is

$$\frac{\partial L_\star(\theta_i,\theta_0)}{\partial\beta} = -2\mu(\theta_i,\theta_0)\cdot\frac{\beta\beta_p}{(\beta-\beta_p)^3}. \tag{42}$$

Since $\mu(\theta_i,\theta_0) > 0$ and $\beta_p < 0$, it can be concluded that

$$\frac{\partial L_\star(\theta_i,\theta_0)}{\partial\beta}\begin{cases} > 0, \ \beta\in(-\infty,\beta_p)\cup(0,+\infty) \\ < 0, \ \beta\in(\beta_p,0)\end{cases}. \tag{43}$$

This completes the derivation.

Appendix B
Solution to (24)

To begin with, it is necessary to show that if $0 \le \rho_k \le 1$ and $\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k) > |\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|$, then

$$\mathbf{B}(2,2) > 0. \tag{44}$$

From the expression of $\mathbf{B}$ in (25), the proof of (44) can be equivalently converted to the following inequality

$$|\mathbf{w}_\parallel^{\mathrm{H}}\mathbf{a}(\theta_k)|^2 > \rho_k|\mathbf{w}_\parallel^{\mathrm{H}}\mathbf{a}(\theta_0)|^2. \tag{45}$$

Owing to the fact that

$$\mathbf{w}_\parallel = \frac{\mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{w}_{(k-1)}}{\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k)} \tag{46}$$

Eq. (45) can be further rewritten as

$$\begin{aligned}\rho_k < \frac{|\mathbf{w}_\parallel^{\mathrm{H}}\mathbf{a}(\theta_k)|^2}{|\mathbf{w}_\parallel^{\mathrm{H}}\mathbf{a}(\theta_0)|^2} &= \frac{\left|\frac{\mathbf{w}_{(k-1)}^{\mathrm{H}}\mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)}{\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k)}\mathbf{a}(\theta_k)\right|^2}{\left|\frac{\mathbf{w}_{(k-1)}^{\mathrm{H}}\mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)}{\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k)}\mathbf{a}(\theta_0)\right|^2} \\ &= \frac{|\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k)|^2}{|\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|^2}.\end{aligned} \tag{47}$$

Obviously, if $\rho_k \le 1$ and $\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k) > |\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|$, we can obtain that

$$\rho_k \le 1 < \frac{|\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_k)|^2}{|\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{a}(\theta_0)|^2}. \tag{48}$$

According to (48), it is readily known that (47) and hence (44) hold true.

To solve the equation (24), we take Eigenvalue decomposition (EVD) of $\mathbf{B}$, i.e.,

$$\mathbf{B} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}} \tag{49}$$

where $\mathbf{Q}$ is an unitary matrix, $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1,\lambda_2\}$ with λ_1 and λ_2 are eigenvalues of $\mathbf{B}$. Define

$$\mathbf{y} \triangleq \mathbf{Q}^{\mathrm{H}}\mathbf{z}. \tag{50}$$

Then (24) can be equivalently expressed as $\mathbf{y}^{\mathrm{T}}\lambda\mathbf{y} = 0$, and further

$$\lambda_1|\mathbf{y}(1)|^2 + \lambda_2|\mathbf{y}(2)|^2 = 0. \tag{51}$$

Since $\lambda_1\lambda_2 = \det(\mathbf{B}) = -\rho_k|\mathbf{w}_\perp^{\mathrm{H}}\mathbf{a}(\theta_0)|^2|\mathbf{w}_\parallel^{\mathrm{H}}\mathbf{a}(\theta_k)|^2 \le 0$, we learn that (51) is logical and can be solved. Denote

$$\mathbf{Q} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{bmatrix}. \tag{52}$$

Since $\mathbf{y} = \mathbf{Q}^{\mathrm{H}}\mathbf{z}$, we have $\mathbf{y}(1) = u_{11}^* + u_{21}^*\beta$ and $\mathbf{y}(2) = u_{12}^* + u_{22}^*\beta$. Then we obtain

$$\begin{aligned}
&\lambda_1|\mathbf{y}(1)|^2 + \lambda_2|\mathbf{y}(2)|^2 \\
&= \lambda_1|u_{11}^* + u_{21}^*\beta|^2 + \lambda_2|u_{12}^* + u_{22}^*\beta|^2 \\
&= \underbrace{\lambda_1|u_{11}|^2 + \lambda_2|u_{12}|^2}_{\mathbf{B}(1,1)} + \underbrace{(\lambda_1 u_{21}^* u_{11} + \lambda_2 u_{22}^* u_{12})}_{\mathbf{B}(1,2)}\beta + \\
&\quad \underbrace{(\lambda_1 u_{11}^* u_{21} + \lambda_2 u_{12}^* u_{22})}_{\mathbf{B}(2,1)}\beta + \underbrace{(\lambda_1|u_{21}|^2 + \lambda_2|u_{22}|^2)}_{\mathbf{B}(2,2)}\beta^2 \\
&= \mathbf{B}(1,1) + 2\Re(\mathbf{B}(1,2))\beta + \mathbf{B}(2,2)\beta^2
\end{aligned} \tag{53}$$

where we have utilized the facts that

$$\mathbf{B}(1,1) = \lambda_1|u_{11}|^2 + \lambda_2|u_{12}|^2 \tag{54}$$

$$\mathbf{B}(1,2) = \lambda_1 u_{21}^* u_{11} + \lambda_2 u_{22}^* u_{12} \tag{55}$$

$$\mathbf{B}(2,1) = \lambda_1 u_{11}^* u_{21} + \lambda_2 u_{12}^* u_{22} \tag{56}$$

$$\mathbf{B}(2,2) = \lambda_1|u_{21}|^2 + \lambda_2|u_{22}|^2. \tag{57}$$

As a result, (51) can be rewritten as a quadratic equation as follows

$$\mathbf{B}(1,1) + 2\Re(\mathbf{B}(1,2))\beta + \mathbf{B}(2,2)\beta^2 = 0. \tag{58}$$

From the expression of $\mathbf{B}$ and the conclusion (44), we have

$$\Re^2(\mathbf{B}(1,2)) - \mathbf{B}(1,1)\mathbf{B}(2,2) \geq 0. \tag{59}$$

Thus, (58) can be analytically solved with solution given by

$$\beta_{a,b} = (-\Re(\mathbf{B}(1,2)) \pm d)/\mathbf{B}(2,2) \tag{60}$$

where $d = \sqrt{\Re^2(\mathbf{B}(1,2)) - \mathbf{B}(1,1)\mathbf{B}(2,2)}$. This completes the derivation.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

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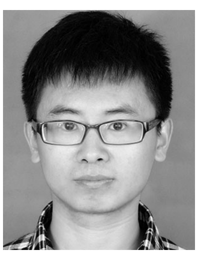

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